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The Endgame

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The Endgame

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Abstract

On December 1st, 2009 President Obama announced that the U.S. troops would have started leaving Afghanistan on July 2011. Rather than simply waiting "the U.S. troops out," the Taliban forces responded with a spike in attacks followed by a decline as the withdrawal date approached. These, at first, counter-intuitive phenomena, are addressed by studying a two-player, zero-sum game where the duration of the strategic interaction is either known or unknown to players. We find that under known duration, players' equilibrium strategies depend on the time remaining in the game and their relative positions at that time of play. Under unknown duration the equilibrium strategies are independent of time and continuation probability. We test the model on data available for soccer matches in the major European leagues. Most importantly, we exploit a change in rule adopted by FIFA in 1998 requiring referees to publicly disclose the length of the added time at the end of the 90 minutes of play. We study how the change in rule has affected the probability of scoring both over time and across teams' relative performance and find that the rule's change led to a 28% increase in the probability of scoring during the added time.

Keywords: conflict resolution, information, soccer.
JEL codes: D74, D83, C72, L83

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1. Introduction

In many instances actors and observers recognize that knowing the exact length of a game-strategic interaction matters, independently of the length itself. By fixing the duration, the parties not only know that the game will end at a certain point in time, but they also know that it will *not* end before then. This is in contrast to the case where the game might be over at *any* point in time. This paper shows that players' equilibrium behavior in a game with a known fixed duration is qualitatively different than in a game with unknown duration.

An army involved in a foreign country intervention is a case in point. The duration of an armed conflict might be either uncertain or fixed. The uncertainty might be simply due to lack of information about how long it would take to resolve the conflict or lack of public or political support. Alternatively, the length of the involvement might be exogenously fixed, *e.g.,* by the budgetary decision of a political body.[1] Regardless of the reason for fixing the length of the involvement, the parties usually recognize that whether the duration is fixed or unknown affects their equilibrium behavior. The Iraq and Afghanistan wars are good examples to demonstrate this point. In both cases the American high command and politicians alike were very much aware of the implications of announcing a definite withdrawal date, as fixing the troops' repatriation essentially fixes the conflict's length, thereby changing the nature of the game from unknown to known duration.[2] In anticipation of a subsequent change in both parties' strategies, the U.S. withdrawal announcement was either preceded by or made contemporary to a surge in troops deployment. Specifically, in the case of Iraq, in preparation of the agreement to hand over to the new Iraqi forces the control of the territory,[3] President Bush ordered a surge in troops in June, 15th 2007. In the case of Afghanistan, President Obama insisted that the announcement of both the troops surge (30,000 troops) as well as the beginning of the withdrawal (July 2011) would occur at the same time.[4] Indeed, both announcements were made during the same speech at West Point on December 1st, 2009 (White House (2009)).[5]

[1]In this paper we will not analyze the case where the duration is part of players' optimal choice.

[2]Among the main points of Senator Obama's first presidential campaign was the setting of a date for troops' withdrawal from Iraq: http://www.washingtonpost.com/wp-dyn/content/article/2007/01/30/AR2007013001586.html.

[3]This was later named the U.S. - Iraq Status of Forces Agreement which fixed the U.S. complete withdrawal to December 31, 2011. This date was later on postponed. For a timeline of the events see http://www.reuters.com/article/2011/12/15/us-iraq-usa-pullout-idUSTRE7BE0EL20111215.

[4]For an account of President Obama's decision of a surge and a withdrawal, see Baker (2009).

[5]This major surge was preceded by an increase in troops of minor entity in February 17, 2009 (17000 troops) and in March 27th, 2009 (4000 troops).

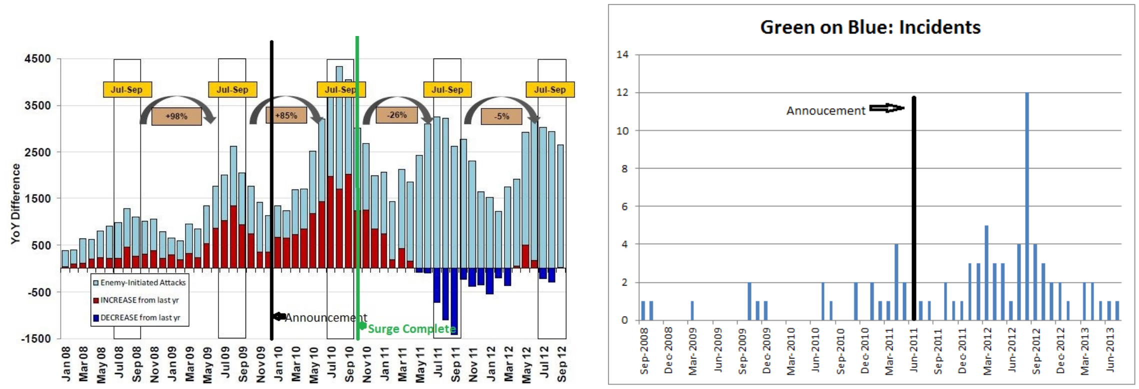
Woodward (2010) reports that President Obama had anticipated a surge in attacks following his West Point speech.[6] Consistent with his expectations, informed observers of the Afghanistan conflict have noticed a discontinuous change in the strategy of the Taliban army in response to Obama's announcement to fix the duration of the involvement of the U.S. forces. The two plots in Figure 1 provide some evidence for these claims. Figure 1(a), published by the NATO's Afghanistan Assessment Group, plots the "Enemy Initiated Attacks" (EIA) by Taliban forces across the period January 2008 - September 2012.[7] Abstracting from seasonality due to the Afghan winter, the figure shows a spike in attacks after the first announcement of troops withdrawals made in November 2009 (Afghanistan Assessment Group (2012)) followed by a gradual decrease in the number of incidents. Figure 1(b) shows the number of attacks on coalition forces by Afghan forces - the so-called "Green-on-Blue" attacks - for the period of September 2008 to June 2013 and includes the date of the second announcement made by the U.S. President (June 22nd, 2011) postponing the U.S. withdrawal to July 2014 along with a troops reduction starting in the following month. The data are consistent with Roggio and Lundquist (2012)'s claim that the number of "Green-on-Blue" attacks "[...] began spiking in 2011, just after President Barack Obama announced the plan to pull the surge forces, end combat operations in 2014, and shift security to Afghan forces. The Taliban also have claimed to have stepped up efforts at infiltrating the Afghan National Security Forces."[8]
These reactions might at first appear counterintuitive. In particular, why did "announcing a timetable for a withdrawal" prompt a surge in attacks by the opponents' army rather than "merely send the Taliban underground until the Americans began to leave," as predicted by Senator McCain in his comment to the West Point speech (PBS (2009))? Similarly, why did President Obama announce a surge in troops concurrently to fixing the duration of the involvement? More generally, why does announcing the duration of the game result in such a discontinuous change in players' behavior?

Armed conflicts are inherently complex. Consequently, we do not attempt a comprehensive rationalization of such intricate events. Rather, we explain why and how knowing versus not knowing

[6] "There is going to be tough, tough fighting in the spring and summer, he added. Anticipate a rise in casualties." (Woodward (2010), p.326). Thanks to Christopher Tuck for pointing this quote out.

[7] In the background (light blue) the total number of EIA. The red bars represent an increase of monthly EIA compared to the same month the year before; blue bars represent a decrease. The changes over three month periods are depicted at the top of the chart. Data Source: Afghan Mission Network (AMN) Combined Information Data Network Exchange (CIDNE) Database, as of 18 Sep 2012.

[8] This claim appears in *The Long War Journal,* among the most comprehensive, available data collection on the attacks conceded by the U.S. troops in the Afghan war.



(a) Year-to-year changes in EIA

(b) Green-on-Blue Attacks Afghanistan

Figure 1: Number of Attacks over Time

the duration of a strategic interaction affects equilibrium actions. We start by modeling a two-player, zero-sum game where players' actions can be classified according to their governance as "attack" or "defense." We assume that all actions require the same level of effort but differ in their probability of success and in the probability of counteracting a successful action by the rival. We study the game under two alternative settings: i) fixed known duration; and ii) unknown duration with a strictly positive probability that at each time of play the game will end in the next period. We characterize the players' optimal strategies which determine the probability of their respective actions to be successful and then study how they change over time and across players' relative positions (*i.e.,* across differences in the number of successful actions).

We find that conditional on the players' relative position, in a known duration game the equilibrium dynamics of the probability of a successful action is non-stationary. The non-stationarity can be monotonic or non-monotonic, depending on the shape of the probability function. In contrast, in an unknown duration game, the dynamics of the probability of a successful action is stationary. Importantly, this implies not only time independence but also independence from the continuation probability (the probability that the game will continue to the next period). This in turn implies that the equilibrium for unknown duration games does not approach the equilibrium for known duration games as the continuation probability is close to one. Still, the probability of a successful action in unknown duration games depends on the players' relative position. We characterize the equilibrium accordingly.

In order to validate the model we exploit the unique opportunity for a natural experiment offered by a change in the rules of the soccer game introduced in 1998 and concerning the time added to the regular time (RT hereafter) in order to recoup any lost minutes during the play, called the added time (AT hereafter). Contrary to the previous regulation, where players did not know with certainty the length of the AT till the referee's whistle was blown, the new rule requires the referee to make the duration of the AT publicly known to players and spectators alike at the end of the RT. In doing so the endgame duration becomes fixed and common knowledge.

We test the prediction of the model on the behavior of the probability of scoring a goal both over time, given the goal differences, and across goal differences, given the time of play. Being the RT a known duration game, we first exploit the availability of the minute-by-minute data for the matches' RT to validate our predictions for known duration games. Consistently with the theoretical model, the empirical analysis shows that during the RT the probability of scoring a goal has an inverted-U shape dynamics over time. For the AT, only data on the total (rather than the minute by minute) number of goals are available. Nevertheless, given the shape of scoring over time in the RT game, we are able to predict the change in rule would increase the number of goals scored during AT. Indeed we find that the average number of goals during the AT is 0.101 and 0.128 for the pre- and post-1998 seasons, respectively, representing a statistically significant increase of 28%. Finally, we find that the probability of scoring a goal during the AT is stationary over the teams' position (in terms of goal difference) .

Our analysis is related to several strands of literature. First is the literature of bargaining games with deadlines (*e.g.,* Spier (1992); and Yildiz (2011)). We argue that knowing the duration of a game is not equivalent to having a deadline as the latter does not prevent the game from stopping beforehand. Consequently, we prefer to adopt the terminology of duration and clearly separate our analysis and results from the deadline effects found in the literature.

There is a large body of work on finite *vs.* infinitely repeated games (*e.g.,* Aumann and Shapley (1994); Rubinstein (1979); and Fudenberg and Maskin (1986)). The strategic situation we study, however, is different than the repeated game setup. Specifically, in our setting the players' final payoff depends on their actions in each period but are not the sum or the average of each period's outcomes. Furthermore, in contrast to the zero-sum nature of our game, the repeated games literature focuses on the incentives to coordinate. These differences explain the contrasting results in our paper and Dal Bó (2005) who studies, in an experimental set up, the equilibrium outcome in a finite *vs.* infinitely repeated prisoner's dilemma game with a random continuation rule. His result showing that the probability of continuation matters for cooperation is driven by the higher expected punishment for deviators when the expected duration of the game is longer.[9]

Our model shares more the features of a sequential tournament, and specifically the case of tournaments where agents choose the level of risk.[10] This literature examines players risk taking as a function of their position in the race. Modelling risk taking, Cabral (2003) finds that leaders choose the safe path while laggards the risky one. This view is also supported by González-Díaz and Palacios-Huerta (2014) in the context of chess tournaments. Hvide (2002) shows that if agents choose both the level of risk and the level of effort, it is possible to limit the risk level agents choose and induce higher effort. While it is easy to frame players' preference in our model in terms of risk and return, this requires additional assumptions on the probability function in order to link the action level to risk taking and would consequently limit the generality of our results.

The literature on the act of sabotage where players expend some of their resources for "the act of raising rivals costs" represents another strand of existing work to which our paper is related.[11] In the Spanish football context, for example, Garicano and Palacios-Huerta (2014) find that increasing the number of points awarded for a win resulted in an increase in the amount of sabotage effort undertaken by teams, measured by the number of fouls, yellow cards and red cards. Allowing players to choose how to allocate their effort between acts that increase one's own probability of success and acts of sabotage adds complexity to the game. Given that, as far as we are aware, this is the first paper to study behavior under known and unknown duration games, we leave the question on the effect of sabotage for further research.

A particularly interesting link is to the literature on demand for suspense (*i.e.,* Eli, Frankel and Kamenica (2013)) . We find that games of known duration display swings in optimal actions between attack and defense. Such swings correspond to higher suspense in known duration *vs.* unknown duration games. Interestingly, this suggests that FIFA's decision to require the communication of the length of the added time has led to two positive outcomes: more goals and more suspense.

[9]See also the literature on repeated trust games, *e.g.,* Engle-Warnick and Slonim (2004).

[10]This is in contrast to the literature on tournaments where agents choose effort levels. See for example, Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983), among many others.

[11]See Chowdhury and Gurtler (2013) for definition and survey of the literature.

2. The game

Consider a game played by two players A and B, over time $t = 1, 2, \ldots$. At each t, each player contemporary chooses an action, $a \in \boldsymbol{A} = [0, 1]$ by player A and $b \in \boldsymbol{B} = [0, 1]$ by B. We interpret higher values of the action as offensive play (or attack) and lower values as defensive play (or defense). At each t players' actions jointly determine the probability of the realization of the random variable $x = -1, 0, 1$ at that t. Specifically, $x = 1$ if player A's action has been successful, $x = 0$ if neither players' action has been successful and $x = -1$ if player B's action has been successful. The probability p_x associated with the realization x is a function of both players' actions and is defined as:

$$p_x : \boldsymbol{A} \times \boldsymbol{B} \to (0, 1), \;\; x = -1, 0, 1. \tag{1}$$

All actions require the same level of effort and bear the same direct costs or disutility. Equivalently, assume they bear no cost.[12] In addition, we assume the following:

Assumption 1. *1. In the interior on the action set, the probability of each player's successful action is increasing in both a and b,* i.e., *for $(a, b) \in Int\boldsymbol{A} \times Int\boldsymbol{B}$:*

$$\partial_a p_x(a, b) > 0 \;\; and \;\; \partial_b p_x(a, b) > 0, \;\; x = -1, 1. \tag{2}$$

2. At the boundaries of the action set, the marginal probability of a successful action by players A and B is such that:

$$\partial_a p_1(1, b) = \partial_a p_{-1}(0, b) = 0; \tag{3}$$

$$\partial_b p_1(a, 0) = \partial_b p_{-1}(a, 1) = 0. \tag{4}$$

3. The probability of a successful action is concave in each player's own action and convex in the opponent's action, i.e.,

$$\partial_a^2 p_1(a, b) < 0 \quad and \;\; \partial_b^2 p_{-1}(a, b) < 0; \tag{5}$$

$$\partial_a^2 p_{-1}(a, b) > 0 \;\; and \;\; \partial_b^2 p_1(a, b) > 0, \tag{6}$$

and the cross derivatives $\partial_{ab}^2 p_x$ and $\partial_{ba}^2 p_x$ are different than 0.

Assumption 1.1 implies that a player choosing a higher action increases both his own and the

[12] Defending typically requires high effort, so there is no direct relationship between the action level and effort level in our model, *e.g.*, action $a = 0$ does not mean inaction. We impose no conditions on the value at $p_x(0, 0)$ and in particular we do not assume that $p_x(0, 0) = 0$, $x = -1, 1$.

opponent's probability of a successful action; the latter representing an implicit cost of attacking. This assumption is meant to capture circumstances where a more offensive action increases the odds of success but weakens the defense level. This trade-off between offense and defense is typical in conflictual situations where one can identify actions of attack or defense. In armed conflicts, for example, an offensive action increases both the chances of inflicting casualties to the enemy and of suffering casualties. In soccer, playing in attack implies an increasing chance of scoring as well as conceding a goal by counter-attack. Assumption 1.2 provides sufficient conditions for obtaining an equilibrium of the game. Assumption 1.3 implies that the marginal probability of a successful action decreases in the player's own action level and increases in the opponent's action level as well as guaranteeing concavity of the players' objective function.

The game is zero-sum. Let d denote the difference in the number of successful actions from player A's perspective. At the end of the game, player A's value from the game is 1 if the difference in successful actions is positive, 0 if nil and -1 if negative, *i.e.,* player A receives a value:

$$V(d) = I\{d \geq 0\} - I\{d \leq 0\}, \tag{7}$$

where I denotes the indicator function. Player B receives a value $-V(d)$.[13]

We can now proceed in analyzing the game when the duration is known, *i.e.,* fixed stopping time, and subsequently (Section 2.2) when the duration is unknown and the game can end at any time with positive probability, *i.e.*, random stopping time. All proofs are in the Appendix.

2.1 Known duration: fixed stopping time

Suppose that both players know that the game will last till $t = T$, *i.e.,* it will end at $t = T$ and not before. Let $V^A(t,d)$ and $V^B(t,d)$ be the values of the game for player A and B, respectively, at time $t < T$, where (t,d) is the state variable identifying the time of play t and the difference in the number of successful actions d at that point in time. Hence, for $t = T$, $V(T,d) = V(d)$. For any $t < T$, the players' optimal actions are the solution to the following system of equations:

$$V^A(t,d) = \max_a \sum_{x=-1}^{1} p_x(a, b^*(t,d)) V^A(t+1, d+x); \tag{8}$$

$$V^B(t,d) = \max_b \sum_{x=-1}^{1} p_x(a^*(t,d), b) V^B(t+1, d+x), \tag{9}$$

[13]These values are for simplicity. The results hold for any increasing function in d that is symmetric around zero.

where $a^*(t,d)$ and $b^*(t,d)$ are the *argmax* of (8) and (9), respectively. The sequence $(a^*(t,d), b^*(t,d) : t = 0, \ldots, T,\ d \in \mathbb{N})$ represents the (Markov perfect) equilibrium of the game of length T.

Notice that given d, the value of the game for player B at $t = T-1$ is given by:

$$\begin{aligned} V^B(T-1,d) &= \sum_{x=-1}^{1} p_x(a^*(T,d), b^*(T,d))V(d+x) \\ &= \sum_{x=-1}^{1} p_x(a^*(T,d), b^*(T,d))(-V(d+x)) = -V^A(T-1,d). \end{aligned}$$

Working backward the same holds for $t = T-2$ and hence at any (t,d), $V^B(t,d) = -V^A(t,d)$. We simplify notation by dropping the superscript and refer to $V(t,d) = V^A(t,d)$ as the *value of the game* at (t,d).

For each t, let us define two *absorbing states* $(t, \overline{d}(t))$ and $(t, -\overline{d}(t))$ where $\overline{d}(t)$ and $-\overline{d}(t)$ are the minimum difference in the number of successful actions necessary for either player A (for $d = \overline{d}(t)$) or player B (for $d = -\overline{d}(t) < 0$) to ensure victory at time t. Specifically, we let $\overline{d}(t) = T - t + 1$. Reaching an absorbing state implies that there is not enough time for the lagging player to catch up or win. Once an absorbing state is reached, the value of the game is fixed either at 1, if player A is winning or at -1 if player B is winning.

Lemma 1. *An equilibrium for the game exists. Specifically, at any (t,d) : 1. if $|d| < \overline{d}(t) - 1$, the equilibrium actions are unique and in the interior,* i.e., $(a^*(t,d), b^*(t,d)) \in Int\boldsymbol{A} \times Int\boldsymbol{B}$;
2. if $d = \overline{d}(t) - 1$ then $(a^*(t,d), b^*(t,d)) = (0,1)$ and if $d = -(\overline{d}(t) - 1)$ then $(a^*(t,d), b^*(t,d)) = (1,0)$;
3. if $|d| \geq \overline{d}(t)$, the equilibrium actions are indeterminate.

Part 1 of the lemma states the existence and uniqueness of the equilibrium when the state of the game is away from the boundaries. Uniqueness also implies that there exists a function $\beta : \boldsymbol{A} \to \boldsymbol{B}$ such that $\beta(a^*) = b^*$ with $\partial_a \beta(a)|_{a=a^*} < 0$. Part 2 identifies the behavior "one successful action away" from an absorbing state. For an intuition of players' behavior in this case consider the state $(t,d) = (T-1, 1)$, *i.e.,* players have only one period left to play and player A is ahead. Player A can choose a relatively offensive action $(a > 0)$ in order to try to increase the probability of state $(T, 2)$; *i.e.,* the game ends with player A leading by 2 successful actions. At the same time, this increases the risk of state $(T, 0)$ where the game ends in a tie. Alternatively, player A can choose a more defensive strategy, for example, set $a = 0$ and maximize the probability of $(T, 1)$. Since $V(2) = V(1) = 1$, and since setting $a = 0$ minimizes the probability of conceding a successful action, the latter strategy is optimal. Similarly, player B can either set $b = 1$ and maximize the probability of $(T, 0)$ (along with increasing the probability of $(T, 2)$) or set $b < 1$ and increase the probability of $(T, 1)$. Since $V(0) < V(1) = V(2)$ the first strategy dominates for player B. Finally, Part 3 of the lemma refers to the case where the game has reached an absorbing state. It is not surprising that the actions are indeterminate in this case as the value of the game cannot be changed while all action levels bear the same cost.

For the remaining part of this section we turn to the characterization of the *interior equilibrium* of a known duration game, *i.e,* at all states (t, d) such that $|d| < \overline{d}(t) - 1$. To this end we to define player A's *relative elasticity of success* for a given action pair (a, b) as the following ratio:[14]

$$\epsilon^A(a, b) = \frac{\partial_a p_1(a, b)/p_1(a, b)}{\partial_a p_{-1}(a, b)/p_{-1}(a, b)}. \tag{10}$$

The relative elasticity of success for player B, $\epsilon^B(a, b)$, can be defined in a similar way.

The variable $\epsilon^A(a, b)$ represents the player's odds of achieving a successful action relative to conceding one. If $\epsilon^A(a, b) > 1$ then an increase in the action by player A, *i.e.,* becoming more offensive, improves the player's relative odds of achieving a successful action as compared to conceding one. Similarly, $\epsilon^A(a, b) < 1$ implies that decreasing the level of A's action, *i.e.,* becoming more defensive, improves the player's relative odds of preventing player B's successful action compared to the odds of achieving one. Accordingly, we say that at (a, b) player A has a *relative advantage in attacking (defending)* if $\epsilon^A(a, b) > 1$ ($\epsilon^A(a, b) < 1$). Similarly for player B.

Let us express the functions of the equilibrium actions $(a^*(t, d), b^*(t, d))$ directly as functions of the state, *e.g.,* $p^*_x(t, d) = p_x(a^*(t, d), b^*(t, d))$ and $\epsilon^{A*}(t, d) = \epsilon^A(a^*(t, d), b^*(t, d))$.

Lemma 2. *At equilibrium, the relative elasticity of success equals the ratio of the expected losses of conceding a successful action to the expected gains of realizing one,* i.e.,

$$\epsilon^{A*}(t, d) = \frac{p^*_{-1}(t, d)\,[V(t + 1, d) - V(t + 1, d - 1)]}{p^*_1(t, d)[V(t + 1, d + 1) - V(t + 1, d)]}. \tag{11}$$

The interpretation of equation (11) is straightforward. A change in action has costs and benefits related to the change in the probability of conceding and realizing a successful action, and accord-

[14] Accordingly, one could define the term $\partial_a p_1(a, b)\frac{a}{p_1(a,b)}$ as player A's elasticity of a successful action and $\partial_a p_{-1}(a, b)\frac{a}{p_{-1}(a,b)}$ as the elasticity of conceding a successful action.

ingly changes the value of the game. In equilibrium, player A sets his action exactly at the point where the relative elasticity of success equals the ratio of the expected marginal costs and benefits. This property will be important for identifying the equilibrium trajectory across d and over t.

By equation (11) it is easy to show that at an interior solution $\epsilon^{A*}(t,d) = [\epsilon^{B*}(t,d)]^{-1}$ and hence that at any given point on the equilibrium trajectory only one player can have a relative advantage in attacking and only one a relative advantage in defending.

Before proceeding to the next lemma, let us denote by $\mathcal{A}_+(b) = \{a : \epsilon^A(a,b) > 1\}$ the set of player A's actions such that, given action b, player A has a relative advantage in attacking. Similarly let $\mathcal{A}_-(b) = \{a : \epsilon^A(a,b) < 1\}$.[15] Also, let $\mathcal{A}^*_+(t,d) = \mathcal{A}_+(b^*(t,d)) = \{a : \epsilon^A(a, b^*(t,d)) > 1\}$.

Lemma 3. *1. At any t the value function is monotonically increasing in d*, i.e., $V(t,d+1) > V(t,d)$.
2. if $a^*(t,d) \in \mathcal{A}^*_+(t,d)$ then given d the value function is monotonically increasing in t, i.e., $V(t+1,d) > V(t,d)$. *The opposite holds for $a^*(t,d) \in \mathcal{A}^*_-(t,d)$.*

Part 1 of the lemma implies that in the interior the marginal value of a successful action is always positive. Part 2 states that "shortening the game", *i.e.,* getting one period closer to the end, has positive marginal value for the player with the relative advantage in attacking. Viceversa for the opponent. The claim follows from rearranging equation (8) to obtain:

$$\overbrace{V(t+1,d) - V(t,d)}^{\text{marginal value of time}} = \overbrace{p^*_{-1}(t,d)[V(t+1,d) - V(t+1,d-1)]}^{\text{expected losses}} - \overbrace{p^*_1(t,d)[V(t+1,d+1) - V(t+1,d)]}^{\text{expected gains}}. \tag{12}$$

The equation shows that the marginal value of time is positive if the expected losses from playing an additional time period are greater than the expected gains. By equation (11) this holds for the player with the relative advantage in attacking. Lemma 3 leads to the following proposition:

Proposition 1. *1. Player A's equilibrium action decreases in d at any given t*, i.e., $a^*(t,d+1) < a^*(t,d)$;
2. If $a^*(t,d) \in \mathcal{A}^*_+(t,d)$, player A's equilibrium action decreases in t at any given d, i.e., $a^*(t+1,d) <$

[15] Notice that, apart for degenerate cases, $\mathcal{A}^*_+(t,d) \cup \mathcal{A}^*_-(t,d)$ is non empty. *E.g.,* the case $\epsilon(t,d) = 1$ for all t can occur when $d = 0$ and when teams are exactly symmetric, *i.e.*, the functional forms of p_1 and p_{-1} are symmetric. Clearly a degenerate case.

$a^*(t,d)$; *The opposite holds if* $a^*(t,d) \in \mathcal{A}^*_-(t,d)$.

Player B's equilibrium action behaves symmetrically.

The result is driven by the assumption that attacking increases the risk of conceding a successful action. In particular, the proposition follows from the fact that the action is inversely related to the value of the game. Across d, since the value of the game increases in d then the action decreases at the same time. The leading player, *i.e.,* the player with higher value of the game, will be more conservative and conversely the losing player more aggressive. Over time, however, this holds only if the expected losses are greater than the expected gains. The player with a relative advantage in attacking chooses a high action at the beginning or, equivalently, reduces his action as the end of the game becomes closer.[16] Viceversa for the other player. [17]

Proposition 1 offers a suggestive interpretation of why Senator McCain's prediction of the Taliban army waiting the U.S. troops out following President Obama's announcement did not materialize. Figure 1.a in the introduction shows a behavior that is consistent with Proposition 1.2. The announcement prompted a spike in Enemy Initiated Attacks followed by a gradual decrease in the subsequent years. Indeed, the announcement represented the beginning of a known duration game. Moreover, if it is reasonable to assume the Taliban had an advantage in attacking in the period right after the announcement then such a response is consistent with the optimal strategy of the player with advantage in attacking in a known duration overtime. This view is supported by Obama's comment on the necessity of breaking the Taliban's momentum with a surge in troops.[18]

2.2 Unknown duration: random stopping time

Suppose now that the players do not know the exact duration of the game and at each time t they assign a probability π_t that the game might continue to the next period, $t+1$. Equivalently, assume that at each t there is a probability $1-\pi_t$ that the game might stop at t. We assume π_t to be strictly positive and less than 1, *i.e.,* letting $\overline{\pi} = \sup \pi_t$ we assume:

Assumption 2. *The continuation probability* π_t *is such that* $0 < \pi_t \leq \overline{\pi} < 1$.

[16]The result is consistent with the empirical observation in the soccer context by Garicano and Palacios-Huerta (2014) where they observe that "[...] when a team is ahead it deploys a strategy aiming at conserving the score relative to the possibility of scoring more goals."

[17]Proposition 1 has very similar properties of a unit root process in t and d of the type: $z_{t-1,d} = \sum_{x=-1}^{1} q_x z_{t,d+x}$ where $\sum_{x=-1}^{1} q_x = 1$ and the terminal condition is given by $z_{T,d} = d$. By backward induction this process has a non-stationary solution for $q_1 \neq q_{-1}$ given by $z^*_{t,d} = (q_1 - q_{-1})(T-t) + d$.

[18]See Woodward (2010), p.329, "[...] 'We have to break the momentum of the Taliban' [Obama] said."

Notice that we allow π_t to be t-dependent, so it needs not be constant and might be decreasing overtime. Also, π_t is bounded away from 0 as this would imply that the game stops with certainty at time t.[19]

Since the stopping time of this game is random this is an infinite horizon stochastic zero-sum game.[20] Let $W^A(t,d)$ ($W^B(t,d)$) denote the value of the game for player A (B) in state (t,d). Recall that once the game ends, the value for player A is given by $V(d)$ and for player B is given by $-V(d)$, where $V(d)$ is as in (7). We can then recursively write the value functions as:

$$W^A(t,d) = \pi_t \max_a \{ \sum_{x=-1}^{1} p_x(a, \tilde{b}(t,d)) W^A(t+1, d+x) \} + (1-\pi_t)V(d); \tag{13}$$

$$W^B(t,d) = \pi_t \max_b \{ \sum_{x=-1}^{1} p_x(\tilde{a}(t,d), b) W^B(t+1, d+x) \} - (1-\pi_t)V(d), \tag{14}$$

where the values $\tilde{a}(t,d)$ and $\tilde{b}(t,d)$ are the equilibrium solutions to (13) and (14).[21] The following proposition characterizes the equilibrium actions for a game of unknown duration.

Proposition 2. *The equilibrium actions of the unknown duration game are t and π_t-independent. Specifically: 1. for $d=0$ they are equal the known duration game equilibrium actions at $T-1$,* i.e., $(\tilde{a}(t,0), \tilde{b}(t,0)) = (a^*(T-1,0), b^*(T-1,0))$.
2. for $d>0$ they are given by $(\tilde{a}(t,d), \tilde{b}(t,d)) = (0,1)$ and for $d<0$ by $(\tilde{a}(t,d), \tilde{b}(t,d)) = (1,0)$.

Proposition 2 has several important implications. First and most importantly the stationarity[22] of the solution implies that removing the certainty about the duration of the game changes players' optimal behavior qualitatively and discontinuously: qualitatively because their behavior becomes stationary; discontinuously because it is independent of the level of uncertainty, *i.e.,* the stopping probability π_t. Mathematically, when the duration is uncertain the contraction mapping theorem holds and we can then identify a stationary solution. The independence from the continuation probability, though surprising, is not different from the standard independence of the policy function from the time discounting in dynamic programming. Similarly, the discontinuity in the action choice

[19] The latter is not equivalent to the known-duration game analyzed in the previous section as that case requires $\pi_t = 1$ for all $t < T$.

[20] *E.g.,* see Parthasarathy and Raghavan (1971).

[21] The tilde distinguishes these solutions from the solutions to (8) and (9) .

[22] Walker, Wooders and Amir (2011) analyze stationary equilibria in unknown duration games. Their analysis focuses on binary Markov games.

from known to unknown duration games is not dissimilar from the well-known problem of analysis of unit roots of AR(1) processes in time series analysis. More intuitively, the action players take at t can affect the game only if it continues to the next period $t+1$ so the continuation value is the only part of the problem that matters, independently of its likelihood (as long as it is greater than zero). By the same token, stationarity follows from the fact that given d, at each t, players look only one period ahead as they know that, given d, they would face the same problem in the following period. That is, given d, the problem is t-independent.[23]

2.3 Characterizing the probability of successful actions

The previous sections have provided a characterization of players' equilibrium actions. However, in many instances only the actions' *outcomes* or *consequences* are observable (*i.e.,* the realization of x) and not the actions themselves. In the case of armed conflicts, records of the attacks (actions) of the armies involved are rarely available and only data on casualties (outcomes) might be recorded. Similarly, in the soccer game, until recently only goals were recorded rather than the players' actions themselves. Nevertheless, by observing outcome x across different d and over time t one may recover the probability of the outcomes, $p^*_x(t,d)$. In this section we show that the results obtained thus far on the equilibrium actions provide testable hypotheses on the behavior of the probability of a successful action *without further restrictions beyond Assumption 1.* Moreover, we show that monotonicity in equilibrium actions does not necessarily translate into monotonicity of the probability of success. In the next section we study a specific functional form of the probability function in order to better characterize its possible trajectories.

We start by looking at changes in the equilibrium probability in known duration games, $p^*_x(t,d)$, over t and across d. Recall that according to Lemma 1.1, player B's equilibrium action can be written as $b^*(t,d) = \beta(a^*(t,d))$. This implies that the equilibrium probability can be written as a function of $a^*(t,d)$ only, *i.e.,* abusing notation $p_x(a^*(t,d))$. In the interior solution of a known duration game, changes to the probability of a successful action due to changes in t and d can be computed as follows:

[23] Proposition 2 can also be explained by looking at well known properties mixed processes of the type $\tilde{z}_{t-1,d} = \pi_t \sum_{x=-1}^{1} q_x \tilde{z}_{t,d+x} + (1-\pi_t)d$. By contraction mapping the process is stationary with solution $\widetilde{z}^*_d = (q_1 - q_{-1}) + d$. The solutions of the process analyzed in footnote 17 and $\widetilde{z}^*_d$ coincide at $t = T-1$ and if $q_1 = q_{-1}$ this holds for all t. Equation (13) describes a mixed process in t with d given by the payoff $W(d)$. The non-stationarity of the dynamics of the solutions to the process in (13) simply follows from the above observation.

$$p_x^*(t,d+1) - p_x^*(t,d) \equiv \Delta_d p_x^*(t,d) \approx \frac{dp_x(a^*)}{da^*}\Delta_d a^*(t,d), \tag{15}$$

$$p_x^*(t+1,d) - p_x^*(t,d) \equiv \Delta_t p_x^*(t,d) \approx \frac{dp_x(a^*)}{da^*}\ \Delta_t a^*(t,d), \tag{16}$$

where Δ_d and Δ_t denote the partial difference with respect to d and t (the approximation is due to the discreteness of t and d). Notice that equations (15) and (16) differ only in the terms Δ_d and Δ_t. Since by Propositions 1 and 2 these are monotonic, non-monotonicities of the probability function are driven uniquely by the non-monotonicity of $\frac{dp_x(a^*)}{da^*}$. The sign of the latter can be determined as:

$$\overbrace{\frac{dp_x(a^*)}{da^*}}^{?} = \overbrace{\partial_a p_x(a^*, \beta(a^*))}^{+} + \overbrace{\partial_b p_x(a^*, \beta(a^*))}^{+}\ \overbrace{\beta'(a^*)}^{-}. \tag{17}$$

The sign of the left hand side is determined by the relative magnitude of the ratio of the two partials $\partial_a p_x(a^*, \beta(a^*))$ and $\partial_b p_x(a^*, \beta(a^*))$ (positive by Assumption 1) and the absolute value of the term $\beta'(a^*)$ (negative by Lemma 1.1). By equation (17) it follows that the equilibrium probability of a successful action is such that:

$$\frac{dp_x(a^*)}{da^*} \geq 0 \text{ if and only if } \frac{\partial_a p_x(a^*, \beta(a^*))}{\partial_b p_x(a^*, \beta(a^*))} \geq -\beta'(a^*). \tag{18}$$

The following result is an immediate corollary of Proposition 1 and 2 along with equations (15), (16) and (18). Denoting by $\tilde{p}(t,d)$ the equilibrium probability in unknown duration games, it follows that:

Proposition 3. *1. In known duration games, $p_x^*(t,d)$ is stationary in t and d if and only if the equality in equation* (18) *is satisfied for all possible a^*.*
2. In unknown duration games, $\tilde{p}_x(t,d)$ is t and π_t-independent. Specifically: a. *for $d = 0$, $\tilde{p}_x(t,0)$ is equal to the known duration game equilibrium probability at $T-1$,* i.e., *$\tilde{p}_x(t,0) = p_x^*(T-1,0)$;* b. *for $d > 0$, $\tilde{p}_x(t,d)$ is given by $\tilde{p}_x(t,d) = p_x^*(T-1,1)$ and for $d < 0$ by $\tilde{p}_x(t,0) = p_x^*(T-1,-1)$.*

Part 1 of the proposition has important implications. Namely, since a turning point in the probability of a successful action is determined by the common term $\frac{dp_x(a^*)}{da^*}$, then the probability of scoring has a turning point in t if and only if has a turning point in d. Part 2 of the proposition states that, in unknown duration games, the probability of a successful action is stationary over time and for $d = 0$. Specifically it is equal to the probability of a successful action one period before the end of a known duration game if $d = 0$ and given by the corner solution in the other cases, where the leading player takes the most defensive action and the losing player the most offensive one. Notice that in the latter case, the probability does not depend on how far ahead or behind a player is such that the probability of a successful action is the same if, for example, the player is 1 successful action or 2 successful actions ahead (behind). This result will be helpful in the empirical validation of the model where we draw indirect inferences regarding the probability of a successful action in unknown duration games when information is available only across d and not over t.

In order to gain a better understanding of how changes in the state (t, d) translate into changes in the probability of a successful action via changes in players' actions, one needs more information, or impose further restrictions, on the actual form of the probability function itself. To this end, in the next section we study a a fairly unrestrictive, yet conveniently simple, class of functions that will help in computing the projection of $p^*(t, d)$ on d and t and hence identify how the probability of observing a successful action might evolve across differences in success and over time.

3. An example

Let us consider the exponentially wrapped log-convex functions:

$$p_1(a, b) = \exp(C_a a - f(a) + f(b)), \tag{19}$$

$$p_{-1}(a, b) = \exp(C_b b - f(b) + f(a)), \tag{20}$$

where C_a and $C_b \in \mathbb{R}_+$, $0 \le f' \le \min\{C_a, C_b\}$ with $f'' > (max(C_a, C_b) - f')^2$, satisfying Assumption 1. All parameters are such that $p_1(a, b) + p_{-1}(a, b) < 1$ for any (a, b).[24] The given functional form allows for the following explicit derivation of $\beta(a^*)$:[25]

$$\beta(a^*) = [f']^{-1} \left[\frac{C_b}{C_a} [C_a - f'(a^*)] \right]. \tag{21}$$

Two facts play a role in the next sections. First, the function $\beta(a^*)$ is a function of f' and hence $\beta'(a^*)$ is a function of f''. It follows that changes in the slope of $p_1(a^*)$ are determined by changes

[24] Alternatively, we could pre-multiply the two functions by a constant small enough to satisfy the inequality. Also note that under this specification if $C_a = C_b$ then the teams are symmetric, *i.e.,* for any any action pair (a, b), $p_1(a, b) = p_{-1}(b, a)$ if and only if $C_a = C_b$. We consider the latter a degenerate case in our setting. This differs from the analysis in the literature which analyzes similar games, *e.g.,* Palomino, Rigotti and Rustichini (1998)

[25] For the derivation of $\beta(a^*)$ see Lemma A1 in Appendix.

in $f^{''}$ and hence by the third derivative of f. Second, using (21), it is easy to see that the ratio of $\partial_a p_1(a^*, \beta(a^*))$ and $\partial_b p_1(a^*, \beta(a^*))$ is C_a/C_b. Thus, changes in $dp_1(a^*)$ are determined by whether $-\beta'(a^*)$ lies above or below C_a/C_b. We can now turn to the characterization of the trajectory of $p_1^*(t, d)$, first across d and then over t.[26]

3.1 The probability of a successful action across d

The following results characterize the trajectory of $\Delta_d p_1^*(t, d)$ for the given functional form.

Proposition 4. *Irrespective of the rules governing the endgame, if $f''' > 0$ ($f''' < 0$) then $p_1(a^*(\cdot, d))$ is inverted-U shaped (U shaped) across d.*

Figure 2 describes the trajectory of the probability of a successful action by player A as a function of his optimal action, given the time of play. The arrows identify the direction of the trajectory as d increases. Let a_p denote the value where (18) holds with equality and let d_p be its projection on d so that a_p is identified by $a^*(\cdot, d_p)$ in the figure. That is at a_p a change in $a^*(\cdot, d)$ is exactly compensated by an opposite change in $\beta(a_p)$ with reaction given by $\beta'(a_p) = -\frac{\partial_a p_1}{\partial_b p_1}$ (for the functional form given in equations (19) and (20) this is equal to $-\frac{C_a}{C_b}$). If d increases from d_p to $d_p + 1$ (since the arrows in the figure point to the left, this corresponds to moving leftward of a_p), by Proposition 1.1 player A's action decreases by an amount, say δ, to $a_p - \delta = a^*(\cdot, d_p + 1) < a^*(\cdot, d_p) = a_p$ and player B's optimal action moves to $\beta(a^*(\cdot, d_p + 1)) > \beta(a^*(\cdot, d_p))$. If $-\beta'(a_p) < -\beta'(a_p - \delta)$ then $p_1^*(\cdot, d_p + 1) < p_1^*(\cdot, d_p)$ and hence $p_1^*(\cdot, d)$ is decreasing (Figure 2(a)).[27] The trajectory is increasing otherwise (Figure 2(b)). Since $\beta'(a_p) - \beta'(a_p - \delta) \approx \delta\beta''(a_p)$, then $-\beta'(a_p) < -\beta'(a_p - \delta)$ if and only if $\beta''(a_p) > 0$ (holding for $f''' < 0$). The same argument applies for decreases in d (moving rightward to a_p). Since a_p for the given functional form is unique (Lemma A2), the behavior of $p_1(a^*(\cdot, d))$ is monotonic thereafter.

3.2 The probability of a successful action over t

As discussed above, in known duration games, the probability of a successful action is non-stationary. In this section, we show that for the functional forms in (19) and (20) the dynamics of

[26] An implication of equation (18) and Lemma 2 is that $\frac{dp_{-1}(a^*)}{da^*} \geq 0$ if and only $\frac{dp_1(a^*)}{da^*} \geq 0$. Hence, $p_1^*(t, d)$ and $p_{-1}^*(t, d)$ have the same behavior across d and over t and we can therefore concentrate on characterizing the behavior of $p_1^*(t, d)$ only.

[27] See Lemma A2.

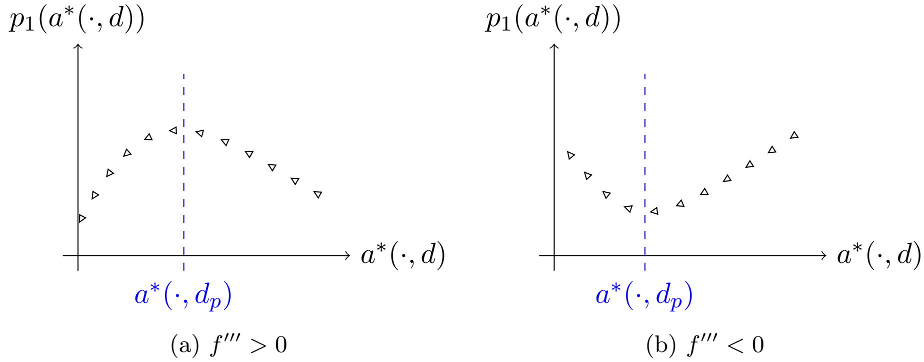


Figure 2: Trajectory of the probability of successful action for increasing d, given t

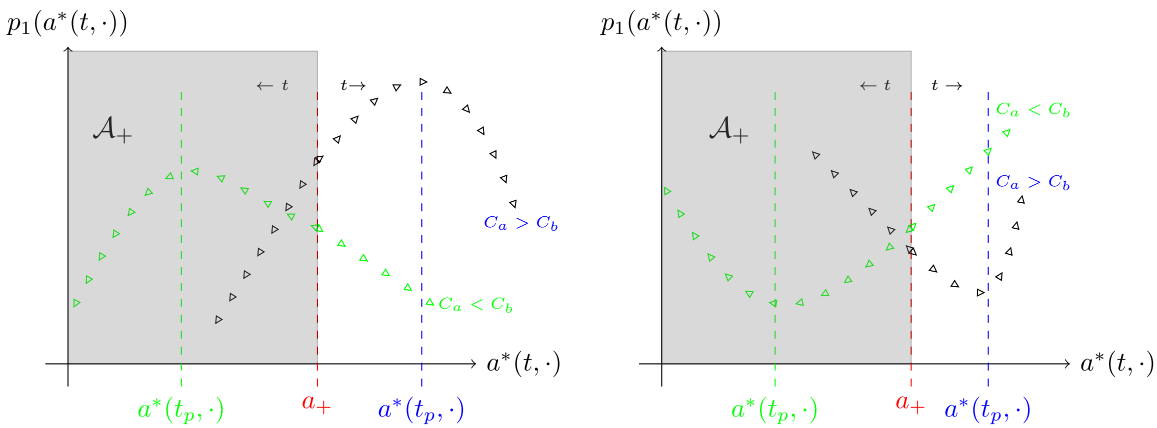
the probability of a successful action depends on the relative position of the two following values along with the shape of the function f. The first value is the turning point $a^*(t_p, \cdot)$, given by the projection of a_p the value where (18) holds with equality on t, where the function $p_1(a^*(t, \cdot))$ is locally concave (convex) if $f''' > 0$ ($f''' < 0$). The second value, denoted by a_+, determines the position of the sets over which player A has a relative advantage in attacking and defending. For the given functional form this is given by $a_+ = [f']^{-1}(C_a/2)$.[28] The point a_+ partitions player A's action space into two connected and (t, d)-independent subsets that can now be written as $\mathcal{A}^*_+$ and $\mathcal{A}^*_-$. The action value $a^*(t, d) \in \mathcal{A}^*_+$ if and only if $a^*(t, d) < a_+$. Moreover the following proposition states that the relative position of $a^*(t_p, \cdot)$ with respect to a_+ depends on the relative magnitude of C_a and C_b.

Proposition 5. *Suppose the duration of the game is known. 1. Let $f''' > 0$. Then, given d, $p_1(a^*(t, \cdot))$ is inverted-U shaped over t on $\mathcal{A}^*_+$ if and only if $C_b > C_a$ or on $\mathcal{A}^*_-$ if and only if $C_b < C_a$. $p_1(a^*(t, \cdot))$ is monotonically decreasing in the complementary sets.*
2. Let $f''' < 0$. Then, given d, $p_1(a^*(t, \cdot))$ is U-shaped over t on $\mathcal{A}^*_+$ if and only if $C_b > C_a$ or on $\mathcal{A}^*_-$ if and only if $C_b < C_a$. $p_1(a^*(t, \cdot))$ is monotonically increasing in the complementary sets.[29]

Figure 3 shows the four qualitatively, non-degenerate[30] configurations of $p_1(a^*(t, d))$ that can occur according to Proposition 5. Graphs (a) and (b) present the cases for $f''' > 0$ and $f''' < 0$, respectively. The light/green and dark/blue arrows trace the dynamics corresponding to $C_b > C_a$ and

[28] See Lemma A3 in the appendix.
[29] Since $p_{-1}(a, b) = p_1(b, a)$, it is possible to compute $p_{-1}(a^*(t, \cdot))$ in a similar way.
[30] The degenerate configurations are for $f''' = 0$ and $C_a = C_b$. The graphs are available from the authors.

$C_b < C_a$, respectively. The shaded area represents the set $\mathcal{A}_+$. The arrows to the left (right) of a_+ show the dynamics over time when the player has a relative advantage in attacking (defending).



(a) $f''' > 0$ (b) $f''' < 0$

Figure 3: Dynamics of the equilibrium probability given d.

Consider, for example, the light/green path in plot (a). This represents the case where $f''' > 0$ and $C_a < C_b$. Since $f''' > 0$, $p_1(a^*(t,d))$ is inverted-U shaped with a turning point at $a^*(t_p,\cdot)$. $C_a < C_b$ then implies that $a_p < a_+$, *i.e.*, a_p belongs to the set of equilibrium actions where the player has a relative advantage in attacking. The mechanism explaining the inverted-U shape of the dynamics over t is the same as for the trajectory over d represented in Figure 2. In this case player A's optimal equilibrium action decreases over t if $a^*(t,\cdot) \in [0; a_+]$. For $a^*(t,d) > a_+$, the optimal action increases overtime and the equilibrium probability $p_1(a^*(t,d))$ decreases with the action resulting in a decreasing probability of successful action. Notice that since the optimal action moves always away from a_+ (*i.e.*, a_+ is a repeller), given d, the optimal action will never cross a_+ overtime and players will not switch relative advantage (unless there is a change in d). If, however, the difference in successful actions increases, $a^*(t,d)$ moves faster away from a_+ if $a^*(t,d) \in \mathcal{A}_+^*$ and it is pushed back towards a_+ if $a^*(t,d) \in \mathcal{A}_-^*$. If the increase in d is sufficiently high then $a^*(t,d)$ crosses a_+ into $\mathcal{A}_+^*$. So, for the case plotted in the green/light path of (a), changes in relative advantage for player A can occur only from defence to attack for sufficiently high increases in d and from attack to defence for sufficiently high decreases in d. The same logic applies to other paths.

Going back to the Afghanistan war context, one can think of casualties as successful actions. Figure

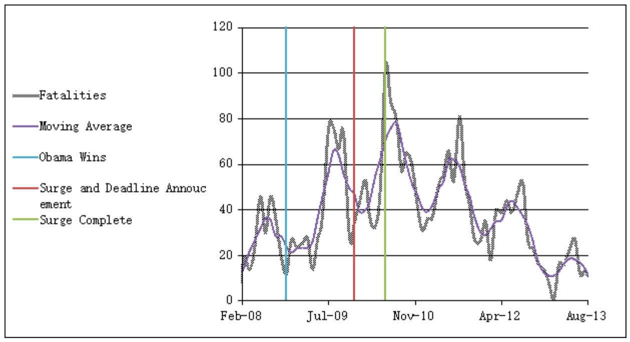


Figure 4: Casualties

4 shows that the trajectory of the allied casualties is non-monotonic. Specifically, the trajectory follows an inverted-U shape overtime. In the next section we compute the equilibrium probability of a successful action for soccer matches and show that, in this case as well, the data are consistent with the inverted-U shaped plots (a) in Figure 3.

4. The endgame in soccer

The Afghanistan war is an interesting example as for the first time one of the belligerent parties publicly announced the duration of the conflict. Unfortunately, although data collections on armed conflicts have substantially improved in the last decades there are still substantial difficulties due to incompleteness and errors.[31] Economic or political competitions present other types of difficulties like complex environments that cannot always be framed in terms of simple models.

This is not the case for sport competitions. There are several well known advantages of using data on sport matches, not only because they are simpler to model than armed conflicts but also because there is a wealth of recorded, relatively reliable data which has shown to be of relevance to strategic analysis well beyond the entertainment industry *per se*.[32] Indeed, for the purpose of this study, sport matches exhibit striking similarities in behavior to armed conflicts like the one in Afghanistan. Among sport matches, we use data from the soccer game. There are two main reasons for this choice. First, soccer is a strategic conflict with a setup similar to our theoretical model. Second and most importantly, a change in the rule governing the added time of the game which requires referees to publicly announce the duration of the added time occurred in 1998. This

[31] For the Afghanistan case, see Cordesman (2012).

[32] See, *e.g.,* Palacios-Huerta (2013) on game theoretic analyses of soccer.

rule change has turned the added time from being of unknown duration to a known duration and thus presents a unique opportunity to test our model.

Unlike other sports, in soccer the game clock is always running and the referee does not pause it for fouls, injuries, penalty kicks, etc. In order to decrease the incentive the leading team might have to waste time so as to stay in the lead, FIFA instructs its referees to add at the end of each half the estimated amount of time lost. The duration of the added or stoppage time is at the sole discretion of the referee, and he/she alone decides when the match is officially over.

Following the characterization of the theoretical part, we can identify two subgames in a soccer match. The first is the regular time (RT), *i.e.,* a known duration game of 90 minutes. The second is the added time (AT) and this can be identified as either of known duration or unknown duration, depending on whether the referee is required to make its length known to the players at the end of the RT subgame.

Originally the duration of the AT was not made public and referees would only blow the whistle to announce the end of the game. This changed in September 1998 when the International Football Association Board (IFAB) required FIFA referees to publicly announce at the end of minute 89 of play the time he/she intends the add to the RT game.[33] Specifically, starting with the 1998-99 season, the referee must communicate the duration of the AT to the fourth official who in turn makes it common knowledge to players and spectators alike by holding up a board reporting the number of minutes to be added to the game.[34] The intention of the change of rule was to make the game more exciting as well as to let players and spectators know that all of the time spent on injuries and other lost time are indeed added back to the game. For the purpose of our analysis, however, the most important effect of the new rule has been to turn the AT from a game of unknown duration, pre-1998 seasons, to one of known duration, post-1998 seasons.

Our theoretical model fits nicely with the settings of the soccer game.[35] Specifically, the RT represents a game of known duration; though the value of the game at the last minutes of play might differ depending on the rule governing the AT. To see this it suffices to interpret the continuation value at minute 89 as the expected length of the added time, either announced or not announced.

[33] See Rule 7 in the FIFA's "Laws of The Game," *http://www.fifa.com.* Our analysis will abstract from the time added to the first 45 minutes.

[34] Starting with the 2011 season the fourth official must keep a written record of the game's interruptions (see http://www.bbc.co.uk/sport/0/football/20159223).

[35] Both armed conflicts and the way to account for wins and losses in soccer are not "pure" zero-sum games. Nevertheless, the data are consistent with our model's predictions.

Our analysis below first examines the RT game. While the RT game is always of known duration, analyzing the RT game serves two objectives. First, it provides a validation of the results of the theoretical model for known duration games. The minute-by-minute data on goals scored allows us to directly estimate the dynamics of the probability of scoring during the RT. Second, both the dynamics and the trajectory across goal differences of the probability of scoring during the RT provide relevant information for comparing the AT games pre- and post-1998, which is the main goal of the empirical analysis. In fact, the analysis of the AT is not straightforward. Unlike the case for the RT, the available record of data on goals scored during the AT consists only of the aggregate scoring, *i.e.,* the total number of goals scored by each team during the entire AT. Although this makes running a direct comparison of the dynamics of the probability of scoring pre- and post-1998 not possible, the aggregate data are still valuable for an indirect inference especially when interpreted together with the behavior of the probability of scoring during the RT.

Our data set is composed of primary-league matches starting with the 1995-1996 season and ending with the 2003-2004 season for England, Germany, Ireland, Italy, Scotland and Spain. For each match, we recorded the total number of goals scored and how far into the game each goal was scored. The data for the analysis were compiled from individual game's box scores, largely obtained from *Soccerbot.com*, an online site reporting results and standings for soccer leagues around the world. In the top division, the English, Italian, German and Spanish leagues have, on average, 20 teams while the Scottish league has an average of 10 teams. In all the leagues studied, each team plays on average 38 games per season, resulting in about 1,500 observations per season.

Table B1 in Appendix B summarizes the number of matches observed, the average total number of goals scored during a game, and the average number of goals scored during AT over the 1995-2003 period.[36]

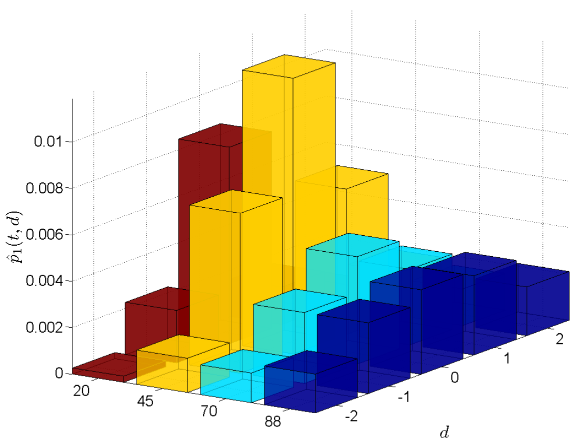
[36] Discussions with sports commentators suggest that during the period we study, it was possible to have some reporting differences where goals scored during AT would later be recorded as scored at minute 89, if the goal was not scored in the last minute of AT. That is, if the game lasted $90 + z$ minutes, then all goals scored between the minute 90 and minute $90 + (z - 1)$ were recorded as having been scored during minute 89. Thus for the empirical analysis, we take the first 88 minutes of each game as RT and the game from minute 90 onward as AT.

4.1 The regular time subgame: a known duration game

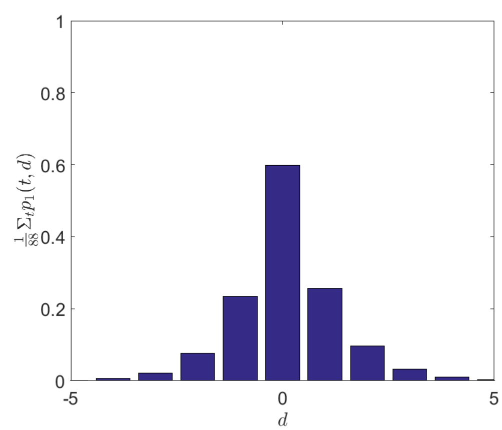
As a first representation of our dataset on RT scoring, Figure 5(a) presents the probability of scoring a goal across d. This is computed as:

$$\hat{p}_1(t,d) = \frac{\#\,(\text{goals scored by team } A \text{ at time } t \text{ when leading by } d)}{\#\text{matches}},$$

where $t = 20$, 45, 70 and 88 is the number of minutes into the game and d is the goal difference by which the team scoring the goal leads or lags. The probability distributions have an inverted-U shape with a turning point d_p around $d = 0$. Figure 5(b) plots the estimated marginal probability of scoring across d by averaging $\hat{p}_1(t,d)$ over time for the RT game. Notice that since $\hat{p}_1(t,d)$ is inverted-U shaped with $d_p(t) = 0$ so is the marginal probability. This can also be observed by summing over t on both sides of equation (15).



(a) $\hat{p}_1(t,d)$ at t=20, 45, 70, 88



(b) $\frac{1}{88}\Sigma_t \hat{p}_1(t,d)$

Figure 5: Estimated probability of scoring across d for given t, graph (a) and averaging for all t, graph (b)

In order to study the shape of the unconditional probability of scoring with respect to both t and d, we fit the following model of the odds of scoring a goal as a function of time and the goal difference:

$$\log(\hat{p}_1(t,d)/(1 - \hat{p}_1(t,d)) = \beta_0 + \delta_1 d + \delta_2 d^2 + \beta_1 t + \beta_2 t^2 + \epsilon, \tag{22}$$

where $t = 1, ..., 88$, is the number of minutes into the game and d is the goal difference.
Our first hypothesis tests for the shape of the model in equation (22).

	Specification I		Specification II		Specification III	
	coeff	z-stat	coeff	z-stat	coeff	z-stat
d	0.0542	2.46	0.0083	0.13		
d^2	-0.5192	-27.97	-0.5206	-27.91		
d^3			0.0137	0.77		
t	0.0530	9.91	0.0648	4.57	0.0100	5.42
t^2	-0.0004	-6.62	-0.0007	-1.95	-8E-05	-3.97
t^3			2.3E-06	0.90		
const	-6.7623	-61.37	-6.8634	-43.45	-4.5365	-127.18
R^2	0.6995		0.7005		0.3997	
F	241.52***		161.01***		28.29***	
Turning point	71				63	

Table 1: Estimation results of the probability ratio for the regular time

Hypothesis 1. $\delta_i = 0,\ i = 1, 2$ *and* $\beta_i = 0,\ i = 1, 2.$

An important implication of Proposition 3 together with equation (15) is that the probability of scoring during RT is non-stationary. In this case, a turning point of $\hat{p}_1(t, d)$ in t implies a turning point in d (and viceversa). Since the odds ratio is a monotonic transformation this should hold for model (22) as well. Rejecting Hypothesis 1 implies that model (22) displays the same behaviour across d and over t. The regression will also allow us to identify the shape of the probability function, a relevant information for the analysis in the next section. In general one can identify three possible paths in the empirical model in (22) over time and goal differences. When $\beta_1 > 0$ and $\beta_2 < 0$ we have an inverted-U shaped curve in t while $\delta_1 > 0$ and $\delta_2 < 0$ imply an inverted-U shaped curve in d. Similarly $\beta_1 < 0$ and $\beta_2 > 0$ imply a U shaped curve in t while $\delta_1 < 0$ and $\delta_2 > 0$ a U shaped curve in d. In the cases where these coefficients have opposite sign, there exists a unique turning point in t (resp. d) at $t_p = -\frac{\beta_1}{2\beta_2}$ (resp. $d_p = -\frac{\delta_1}{2\delta_2}$). Finally if both β's coefficients (resp. δ's) have the same sign, the function is monotonic in t (resp. d), either increasing or decreasing. The behavior of $\hat{p}_1(t, d)$ over t and across d must be the same, non-monotonic behavior if we fail to reject the following hypothesis:

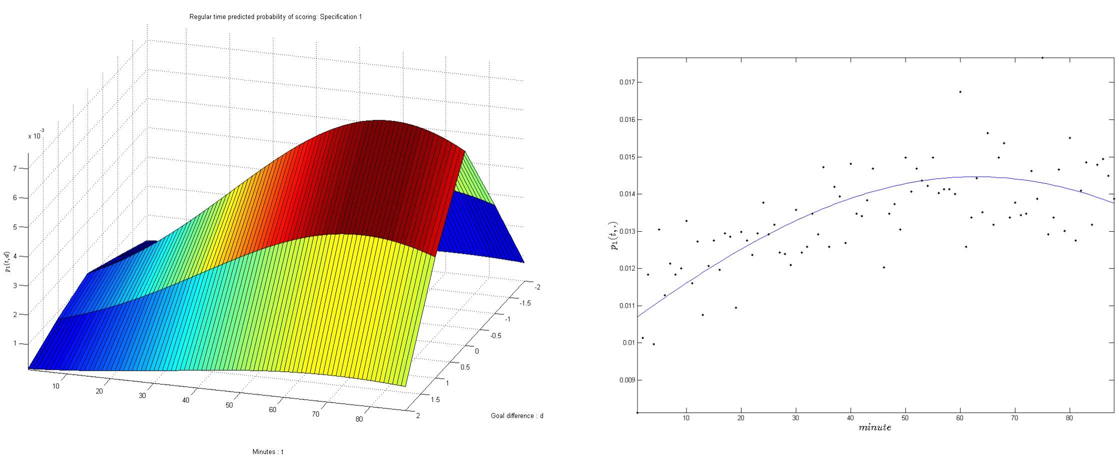
Hypothesis 2. *Either:* $\beta_1 > 0,\ \delta_1 > 0$ *and* $\beta_2 < 0,\ \delta_2 < 0,$
or: $\beta_1 < 0,\ \delta_1 < 0$ *and* $\beta_2 > 0,\ \delta_2 > 0.$

Table 1 presents the estimation results for the RT game. Specification I summarizes the estimation of model (22). Specification II overfits the model with cubic terms to check for parsimony and whether there is more than one turning point. Specification III estimates the marginal probability over t without controlling for d (analogous to Figure 5(b)).

The results in Specification I show that we reject Hypothesis 1. We also find the coefficients on t and d are both positive and significant while the coefficients on t^2 and d^2 are both negative and significant, which confirms that the probability of scoring has an inverted-U shape in t and d. Hence, we fail to reject Hypothesis 2. The turning points occur at $t_p = 71$ and $d_p = 0.52$. Figure 6 (a) plots the response surface for visual confirmation.

Specification II rejects a cubic relationship hence validating Specification I. Though the purpose of the analysis is not to test whether the probability function fits with the probability family presented in (19) and (20) in the example, the results imply that we cannot reject this hypothesis.

Finally, Specification III shows that the inverted-U relationship remains even when we do not control for goal differences. The comparative response curve is plotted in Figure 6(b).



(a) Specification I

(b) Specification III

Figure 6: The dynamics of the probability of scoring during RT

4.2 The added time subgame: a natural experiment

The 1998 rule change governing the AT represents an opportunity to run a natural experiment to test our model. As already pointed out, a direct comparison of the dynamics of the probability of scoring before and after the rule change is not possible due to the lack of minute-by-minute data but an indirect inference is possible by using the analysis on the RT games in conjunction with the behavior of the probability of scoring across d during the AT games.

Two notes are in order. First, since the teams playing in the RT and in the AT games are the same, we assume that the function characterizing the probability of scoring in the RT and in the AT games is the same. Second, in the lack of contrary evidence, we assume the same average length of the AT before and after 1998 and hence that there is no significant marginal change in the scoring probability due to any extra time.[37]

We first test that in the post-1998 games the average number of goals scored during AT, conditional on the goal difference at the beginning of AT, have the same inverted-U trajectory in d as in the RT game. Having disaggregated data on d but not over t, the behavior across d is useful for inferring the behavior over t. This follows from the implication of Proposition 3.1, *i.e.,* if the probability of scoring in known-duration games has a turning point in d then it has a turning point in t. To this end we estimate the following logit model of the probability of scoring a goal during AT post-1998, conditional on d:

$$y_i = f\left(\gamma_0 + \gamma_1 d_i + \gamma_2 d_i^2\right) + \epsilon_i, \tag{23}$$

where y_i equals 1 if a goal was scored in the i^{th} game and zero otherwise,[38] and f denotes the logit link function. Since the data is aggregated over time, the model is t-independent.

In order for the data to be consistent with Proposition 3.1 the coefficients γ_1 and γ_2 in the logit model (23) applied to post-1998 AT games must have the same sign as the coefficients δ_1 and δ_2 in model (22), respectively. The following hypothesis formalizes this observation:

Hypothesis 3. $\gamma_1 > 0$ *and* $\gamma_2 < 0$.

The results for post-1998 AT games reported in Table 2 show that the coefficients on γ_1 and γ_2 are positive and negative, respectively. This implies that model (23) for the post-1998 AT games is an inverted-U shaped curve in d as in the case of the probability function across d during the RT game. The statistical insignificance of the cubic term of d further validates our hypothesis. Table 2 also reports the results for pre-1998 games.

We can now look at the implications of the change in rule on the probability of scoring over time. Lacking minute-by-minute data, the argument we use for the indirect inference makes use of the

[37]This has been confirmed in discussions with sports commentators.

[38]We exclude games with multiple goals during AT, so that the conditionality on d does not change.

	Pre-1998				Post-1998			
	coeff	Z-stat	coeff	z-stat	coeff	z-stat	coeff	z-stat
d	0.1427	4.50	0.1589	3.61	0.0824	4.77	0.1184	4.78
d^2	-0.0184	-1.66	-0.0162	-1.39	-0.0155	-2.42	-0.0142	-2.25
d^3			-0.0017	-0.54			-0.0037	-2.06
cons	-2.9103	-50.28	-2.9166	-49.39	-2.6480	-81.81	-2.6506	-81.83
LL	-1697.26		-1697.12		-5208.84		-5206.85	
χ^2		22.95		31.56		27.59		23.23

Table 2: The probability of scoring a goal with respect d during AT

behaviour of the probability of scoring in the RT games across d and t and in the AT across d. Specifically, notice that the continuation value of the RT game at the last minute of play is given by the expected value of the AT game. Furthermore, recall that according to Proposition 3.2, the probability of scoring a goal in an unknown duration game at $d \leq -1$, $d = 0$ and $d \geq 1$ equals the probability of scoring at the last minute of the known duration game at $d = 1$, $d = 0$ and $d = 1$, respectively. Taken together, this suggests that for the pre-1998 seasons, since the AT game is an unknown duration game, the dynamics of probability of scoring during AT is time invariant and should equal the "end point" of the probability of scoring during AT in the post-1998 games. For the post-1998 seasons, the expected value is taken over all possible AT lengths the referee may announce at the end of the RT game.[39] Being a continuation of the RT game, it follows that the probability of scoring during AT in the post-1998 games is a downward sloping curve. This together with the fact that at the end of the AT the probability of scoring during AT pre-1998 and post-98 must be the same implies that, for any given d, the probability of scoring during AT post-1998 is always above the corresponding probability in the pre-1998 games. Consequently, the average number of goals scored during AT post-1998 should be greater than this average pre-1998. Figure 7 illustrates the argument.

The area between the curves in the figure represents the increase in the probability of scoring during AT due to the change of rule. To this end we estimate the following model:

$$y_i = f\left(\tilde{\gamma}_0 + \tilde{\gamma}_{0,d} D98_i + \tilde{\gamma}_1 d_i + \tilde{\gamma}_2 d_i^2\right) + \epsilon_i. \tag{24}$$

The dummy variable $D98$ equals 1 if the game is played after the rule change and 0 otherwise; d

[39] While this expectation might over or under-estimating the actual time added, it is reasonable to assume that on average players correctly anticipate the referee's announcement.

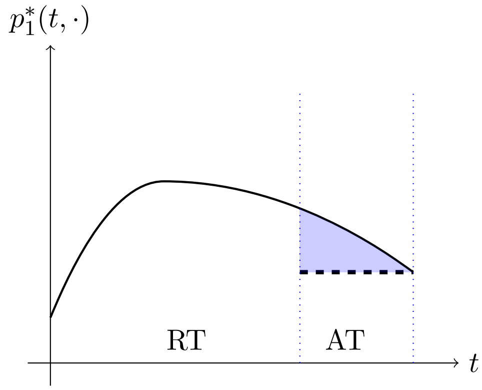


Figure 7: Probability of successful action

	Specification I	
	coeff	z-stat
d	0.0969	6.40
d^2	-0.0159	-2.87
D98	0.2559	4.52
cons	-2.9039	-56.10
LL	-6907.55	
χ^2	69.16	

Table 3: Logit estimation

is the goal difference at the beginning of the AT game. The coefficient $\tilde{\gamma}_{0,d}$ measures the average goal difference pre and post-1998 and hence the highlighted area in Figure 7. We can state our hypothesis as follows:

Hypothesis 4. *The coefficient* $\tilde{\gamma}_{0,d} > 0$.

Specification I in Table 3 summarizes the results of equation (24). The z-stat value confirms that the coefficient on $\tilde{\gamma}_{0,d}$ is positively significant. Thus we accept Hypothesis 4.

Most interestingly, the results show that the change of policy on the disclosure of the duration of the added time game has led to a 28% increase in the probability of scoring from 0.101 to 0.128 (see Table B1), implying more than one extra goal every 4 games.

Our final test analyzes the prediction of Proposition 3.2 on the dynamics of the probability of a successful action in unknown duration games, namely that the probabilities of scoring are constant across $d < 0$ and across $d > 0$. Due to the small number of observations, we cannot test this predictions for observations where $d < -2$ and $d > 2$. Therefore we test whether the probability of scoring at $d = -2$ ($d = 2$) equals the probability of scoring at $d = -1$ ($d = 1$). The following hypothesis formalizes the test:

Hypothesis 5. $\tilde{p}(-2) = \tilde{p}(-1)$ *and* $\tilde{p}(1) = \tilde{p}(2)$.

Estimating the conditional probabilities for $\tilde{p}(-2)$ and $\tilde{p}(-1)$ we obtain 0.051 and 0.040, respectively with a z-stat of -1.24. Similarly, for the conditional probabilities for $\tilde{p}(1)$ and $\tilde{p}(2)$ we obtain 0.072 and 0.071, respectively with a z-stat of 0.14. We conclude the analysis by accepting Hypothesis 5.

5. Concluding remarks

The 2008 U.S. presidential campaign, where the setting of a withdrawal date from Afghanistan was central to the debate, provides evidence of policymakers' awareness of the potential implications of disclosing the duration of a conflict, both as a response to public opinion pressure and as a strategic commitment. A better understanding of these issues would also help in the management of international peace-keeping missions, especially when considering the optimal allocation of troops across multiple fronts.

Further work is necessary for a full assessment of how the two games analyzed in this paper differ. Among them, notice that Table B1 reports the statistics for the average total number of goals throughout the game across the different seasons. It is interesting to notice that the communication of time left till the end of the game has not affected the average total number of goals scored in a match. Rather, it has significantly reallocated goals from the RT subgame toward the AT subgame. We did not try to address the reason for the scoring reallocation as this will be the objective of future work. However, the fact that the total number of goals did not increase over time has very important implications as it suggests that the increase in goals scored during AT cannot be simply attributed to an increase in the average duration of the AT (of which, moreover, there is no evidence) as this would have affected the total number of goals scored in a game.

As a final note, in our theoretical model and in the empirical application, neither the game's duration nor the communication of the duration is part of the players' strategies, as both are taken to be exogenous. An interesting extension of the model would consider the case where agents can unilaterally fix the duration and then decide whether to release this information or keep it private. Notice that having abstracted away from this case does not detract from the interest of the analysis as in many situations the duration of the game is not part of the players actions' set. For example, in the case of armed conflicts or peace missions, budgetary and political considerations often determine the length of the involvement, which is only then communicated to the actors on the field. In case of the UN peace missions the "The Fifth Committee (Administrative and Budgetary) sets the Peacekeeping Budget each year from July to June. However, the committee reviews and adjusts the budget throughout the year. Since peace missions vary in number and duration, contributions to the Peacekeeping Budget fluctuate widely from year to year" (Global Policy Forum (2014)).

References

Afghanistan Assessment Group (2012), "International Security Assistance Force in Afghanistan Monthly Data - Trends through February 2012." *http://www.isaf.nato.int/images/20120320_isaf_ monthly_data.pdf*

Aumann R.J. and L.S. Shapley (1994), *Essays in Game Theory,* Springer New York.

Baker, P. (2009), "How Obama came to plan for "surge" in Afghanistan", *New York Times,* 5/12/2009.

Cabral, L.M.B. (2003),"R&D competition when firms choose variance," *Journal of Economics and Management Strategy*, 12(1), 139-150.

Chowdhury S.M. and Gurtler O. (2013), "Sabotage in contests: a survey," University of East Anglia Applied and Financial Economics Working Paper Series 051.

Cordesman, A.H. (2012), "Afghanistan: the failed metrics of ten years of war," Report 120209, Center for Strategic and International Studies.

Dal Bó P. (2005), Cooperation under the shadow of the future: experimental evidence from infinitely repeated games, *American Economic Review*, 95(5) 1591-1604.

Eli, J., A. Frankel and E. Kamenica (2013), "Suspense and surprise," mimeo.

Engle-Warnick, J. and R. Slonim (2004). "The evolution of strategies in a repeated trust game." *Journal of Economic Behavior and Organization,* 55(4), 553-573.

Fundeberg, D. and E. Maskin (1986), The folk theorem in repeated games with discounting and with incomplete information, *Econometrica*, 54(3), 533-554.

Garicano L. and I. Palacios-Huerta (2014), "Sabotage in tournaments: making the beautiful game a bit less beautiful," in Palacios-Huerta (2014).

González-Díaz, J. and I. Palacios-Huerta (2014)"Cognitive performance in competitive environments: evidence from a natural experiment."

Global Policy Forum (2014), "Tables and charts on UN Peacekeeping Operations Budget," *http://www.globalpolicy.org/un-finance/tables-and-charts-on-un-finance/the-un-peacekeeping-operations-budget.html.*

Hvide, H.K. (2002), "Tournament rewards and risk taking," *Journal of Labor Economics,* **20**(**4**), 945-976.

Lazear E.P. and S. Rosen (1981), "Rank-order tournaments as optimum labor contracts," *Journal of Political Economy*, 89(5), pp. 841-864.

Nalebuff, B.J. and Stiglitz, J.E. 1983, "Prices and incentives: towards a general theory of compensation and competition," Bell Journal of Economics,14(1), 21-43. Palacios-Huerta, I. (2014), *Beautiful Game Theory,* Princeton University Press.

Parthasarathy T. and T.E.S. Raghavan (1971), *Some topics in two-person games,* Elsevier.

PBS (2009), "Obama outlines plan for Afghanistan troop surge," http://www.pbs.org/newshour/up dates/military-july-dec09-obamapeech_12-01/

Palomino, F. A., Rigotti, L. and Rustichini, A. (1998), “Skill, strategy and passion: an empirical analysis of soccer.” *CentER Discussion Paper*; Vol. 1998-129.

Roggio, B. and L. Lundquist (2012), “Green-on-blue attacks in Afghanistan: the data,” *The Long War Journal,* http://www.longwarjournal.org/archives/2012/08/green-on-blue_attack.php.

Rubinstein, A. (1979), “Equilibrium in supergames with the overtaking criterion,” *Journal of Economic Theory,* **21**(1), 1-9.

Spier, K.E. (1992), “The Dynamics of pretrial negotiation,” *The Review of Economic Studies,* **59**(**1**) 93-108.

Walker, M. J. Wooders, and R. Amir (GEB (2011)), “Equilibrium play in matches: binary Markov games,” *Games and Economic Behaviour,* **71**, 487-502.

White House, Office of Press Secretary (2009), *Remarks by the president in address to the nation on the way forward in Afghanistan and Pakistan,* http://www.whitehouse.gov/the-press-office/remarks-president-address-nation-way-forward-afghanistan-and-pakistan.

Woodward, B. (2010) *Obama's Wars*, Simon & Schuster.

Yildiz, M. (2011), “Bargaining with optimism,” *Annual Review of Economics,* **3** 399-420.

Appendix A

Proof of Lemma 1: Consider the following function of $(a, b) \in \mathbf{A} \times \mathbf{B}$:

$$U(a, b; \mathbf{V}) = \sum_{x=-1}^{1} p_x(a, b) V_x, \tag{A1}$$

where $\mathbf{V} = (V_x : x = -1, 0, 1)$ and $V_1 \geq V_0 \geq V_{-1}$ represents a vector of parameters and the probability function $p_x(\cdot, \cdot),\ x = -1, 0, 1$ satisfies Assumption 1. Let:

$$a^* = \arg\max_a U(a, b^*; \mathbf{V}), \tag{A2}$$

$$b^* = \arg\min_b U(a^*, b; \mathbf{V}). \tag{A3}$$

We will later interpret a^* and b^* as $a^*(t, d)$ and $b^*(t, d)$, respectively. Consider the following three cases that we will subsequently relate to Part 1-3 in the lemma.

Case A1: $V_1 > V_0 > V_{-1}$. A necessary condition for an interior equilibrium is that for any given

$b \in (0,1)$: $\partial_a U(a,b;\mathbf{V}) = 0$. For all $\mathbf{V}$ by Assumption 1.2 it follows that:

$$\partial_a U(0,b;\mathbf{V}) = \partial_a p_1(0,b)(V_1 - V_0) - \partial_a p_{-1}(0,b)(V_0 - V_{-1}) = \partial_a p_1(0,b)(V_1 - V_0);$$
$$\partial_a U(1,b;\mathbf{V}) = \partial_a p_1(1,b)(V_1 - V_0) - \partial_a p_{-1}(1,b)(V_0 - V_{-1}) = -\partial_a p_{-1}(1,b)(V_0 - V_{-1}).$$

By Assumption 1.3, $\partial_a^2 p_1(a,b) < 0$ and hence $\partial_a p_1(0,b) > \partial_a p_1(1,b) = 0$. Also, since by the same assumption $\partial_a^2 p_{-1} > 0$ we have $\partial_a p_{-1}(1,b) > \partial_a p_{-1}(0,b) = 0$. Thus $\partial_a U(0,b;\mathbf{V}) > 0$ and $\partial_a U(1,b;\mathbf{V}) < 0$. By the intermediate value theorem given $b \in (a,b)$ there exists a value $a^*(b) \in (a,b)$ such that $\partial U(a^*(b),b;\mathbf{V}) = 0$. A similar argument proves that given $a \in (a,b)$ there exists a value $b^*(a) \in (a,b)$ such that $\partial U(a,b^*(a);\mathbf{V}) = 0$.

In order to show uniqueness, notice that by implicit function theorem, the slope of the reaction function is given by:

$$\frac{da^*(b)}{db} = -\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a \partial b}\left(\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a^2}\right)^{-1}.$$

Similarly, again by the implicit function theorem on the first order condition of the dual problem $\min_b U(a,b;\mathbf{V})$ for any given a obtain:

$$\frac{db^*(a)}{da} = -\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a \partial b}\left(\frac{\partial^2 U(a,b;\mathbf{V})}{\partial b^2}\right)^{-1}.$$

Notice that by assumption $\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a^2} < 0$ and $\frac{\partial^2 U(a,b;\mathbf{V})}{\partial b^2} > 0$. Having the numerators the same signs, the slopes of the reaction functions $\frac{da^*(b)}{db}$ and $\frac{db^*(a)}{da}$ have opposite signs. Thus the value at which the reaction functions cross is unique. This also shows that there exits a function $\beta : \mathbf{A} \to \mathbf{B}$ such that $\beta(a^*) = b^*$ and $\beta' < 0$.

Case A2: This comprises two sub-cases. The first where $V_1 = V_0 > V_{-1}$. The problem becomes:

$$\begin{aligned} U(a,b;\mathbf{V}) &= (1 - p_{-1}(a,b))\, V_0 + p_{-1}(a,b) V_{-1}, \\ &= V_0 - p_{-1}(a,b)\,(V_0 - V_{-1})\,. \end{aligned} \tag{A4}$$

Thus for any given b^*, maximising $U(a,b^*;\mathbf{V})$ with respect to a, implies choosing the minimum value of $a^* = 0$. Similarly, for any given a^* minimising $U(a^*,b;\mathbf{V})$ with respect to b, implies choosing the maximum value of $b^* = 1$. Then $(a^*,b^*) = (0,1)$.

The second sub-case occurs for $V_1 > V_0 = V_{-1}$. A similar argument obtains $(a^*,b^*) = (1,0)$.

Case A3: $V_1 = V_0 = V_{-1}$. This implies $U(a, b; \mathbf{V}) = V_0$ and then any (a^*, b^*) is a solution.

In order to conclude the proof it suffices to reinterpret a^* and b^* as the solutions $a^*(t, d)$ and $b^*(t, d)$ to the problems in (8) and (9), respectively, where $V(t+1, d+1)$ is equivalent to V_x in the static problem.

Finally, the cases $d < |T - t + 1| - 1$, $d = |T - t + 1| - 1$ and $d \geq |T - t + 1|$ correspond to Case A1, A2 and A3 above and to point 1, 2 and 3 in the lemma. □

Proof of Lemma 2: From the first order condition of (8) computing the derivatives at $a^* = a^*(t, d)$ and $b^* = b^*(t, d)$, recalling that the game is zero-sum and using $p_0 = 1 - p_1 - p_{-1}$ obtain:

$$\partial_a p_1(a^*, b^*)[V(t+1, d+1) - V(t+1, d)] = \partial_a p_{-1}(a^*, b^*)[V(t+1, d) - V(t+1, d-1)], \qquad \text{(A5)}$$

Rearranging (A5) and multiplying both sides by $\frac{p^*_{-1}(t,d)}{p^*_1(t,d)}$ obtain (11). $\epsilon^{B^*}(t, d)$ can be obtained by rearranging the first order condition of (9) in a similar way. □

Proof of Lemma 3. *Part 1.* From the first order condition of (8) evaluated at equilibrium we obtain:

$$V(t+1, d) - V(t+1, d-1) = \frac{\partial_a p^*_1(t, d)}{\partial_a p^*_{-1}(t, d)}[V(t+1, d+1) - V(t+1, d)]. \qquad \text{(A6)}$$

where $\partial_a p^*_x(t, d) > 0$ by Assumption 1.1. From Equation (A6) it follows that $V(t+1, d+1) - V(t+1, d)$ and $V(t+1, d) - V(t+1, d-1)$ must have the same sign and hence that $V(\cdot, d)$ is a monotone function in d. Since at the absorbing states $V(\cdot, -\bar{d}(\cdot)) = -1$ and $V(\cdot, \bar{d}(\cdot)) = 1$, it follows that $V(\cdot, d)$ must be increasing in d.

Part 2. See the main text. □

Proof of Proposition 1: Consider the function in (A1) and let $U(a^*, b^*; \mathbf{V}) = \max_a U(a, b^*; \mathbf{V})$. By the envelope theorem it follows that:

$$dU(a^*, b^*; \mathbf{V}) = \sum_{x=-1}^{1} p_x(a^*, b^*) \partial V_x.$$

Holding the value of $U(a^*, b^*; \mathbf{V})$ constant (*i.e., $dU(a^*, b^*; \mathbf{V}) = 0$*) and holding $\partial V_{-1} = 0$, obtain: $\frac{\partial V_1}{\partial V_0} = -\frac{p_0(a^*,b^*)}{p_1(a^*,b^*)}$. Similarly holding $\partial V_1 = 0$, obtain: $\frac{\partial V_{-1}}{\partial V_0} = -\frac{p_0(a^*,b^*)}{p_{-1}(a^*,b^*)}$. Using the implicit function theorem on the first order condition:

$$\frac{da^*}{dV_0} = -\frac{\partial^2 U\left(a, b^*; \mathbf{V}\right)}{\partial a \partial V_0}\left(\frac{\partial^2 U\left(a, b^*; \mathbf{V}\right)}{\partial a^2}\right)^{-1}.$$

By Assumption 1.3 the function U is concave and hence $\frac{\partial^2 U(a^*(\mathbf{V}), b^*(\mathbf{V}))}{\partial a^2} < 0$. Moreover:

$$\begin{aligned}\frac{\partial^2 U\left(a, b^*; \mathbf{V}\right)}{\partial a \partial V_0} &= \partial_a p_1^* \frac{\partial V_1}{\partial V_0} + \partial_a p_{-1}^* \frac{\partial V_{-1}}{\partial V_0} + \partial_a p_0^* \frac{\partial V_0}{\partial V_0} \\ &= -\partial_a p_1^* \frac{p_0^*}{p_1^*} - \partial_a p_{-1}^* \frac{p_0^*}{p_{-1}^*} - \partial_a p_1^* - \partial_a p_{-1}^* < 0,\end{aligned}$$

where the inequality follows by Assumption 1.1. This shows that $\frac{da^*(\mathbf{V})}{dV_0} < 0$. A similar argument shows that $\frac{db^*(\mathbf{V})}{dV_0} > 0$.

The above static formulation of the problem is convenient as now we can prove the two parts of the proposition by appropriately reinterpreting the values V_x.

Part 1. Recall that by Lemma 3.1, the value of the game is increasing in the differences of successful actions. Consider now two different values for V_0 in problem (A1), namely $V_0^{'} = V\left(t, d\right)$ and $V_0^{''} = V\left(t, d+1\right) > V\left(t, d\right) = V_0^{'}$. It follows that the solution $a^*\left(t, d\right)$ to (A1) when $V_0 = V_0^{'}$ and the solution $a^*\left(t, d+1\right)$ when $V_0 = V_0^{''}$ are such that $a^*\left(t, d+1\right) < a^*\left(t, d\right)$.

Part 2. Recall that by Lemma 3.2, if $a^* \in \mathcal{A}_+(t, d)$ then $V\left(t+1, d\right) > V\left(t, d\right)$. Suppose player A has advantage in attacking and consider two different values for V_0 in problem (A1), namely $V_0^{'} = V\left(t, d\right)$ and $V_0^{''} = V\left(t+1, d\right) > V\left(t, d\right) = V_0^{'}$. It follows that the solution $a^*\left(t, d\right)$ to (A1) when $V_0 = V_0^{'}$ and the solution $a^*\left(t, d\right)$ when $V_0 = V_0^{''}$ are such that $a^*\left(t+1, d\right) < a^*\left(t, d\right)$.

Being the game zero-sum, the behaviour of player B's optimal action is symmetric. □

Proof of Proposition 2: The proof proceeds in several steps.

Step 1. We start by showing that the solution to problem (13) and (14) is t-independent. Let:

$$W(d) = \pi \sum_{x=-1}^{1} \tilde{p}_x(d) W'(d+x) + (1-\pi) V(d) = \Phi(W')(d),$$

where Φ is the continuous functional $\Phi : \Upsilon \to \Upsilon$ with Υ the space of bounded functions endowed with the uniform norm $\|\cdot\|_\infty$. Consider W and $W' \in \Upsilon$, then

$$\left\|\Phi(W) - \Phi(W')\right\|_\infty \leq \pi \left\|W - W'\right\|_\infty \leq \overline{\pi} \left\|W - W'\right\|_\infty.$$

Since $\overline{\pi} < 1$, Φ is a contraction mapping with fixed point in Υ, *i.e.,* the solution is t-independent.

Step 2. We now show two facts that will be useful in Step 3 to characterize the solution of the system. First we prove that for all d's, $|W(d)| < 1$; then we prove that $W(1) > W(0) > W(-1)$. In order to prove that $|W(d)| < 1$, rewrite (13) at the t-independent solution:

$$W(d) = \pi\left[\tilde{p}_1(d)\,W(d+1) + \tilde{p}_0(d)\,W(d) + \tilde{p}_{-1}(d)\,W(d-1)\right] + (1-\pi)V(d)\,. \tag{A7}$$

The sequence of equations represented by (A7) for $d = \dots, -1, 0, 1, \dots$ can be rewritten as the linear functional $\mathbf{W} = \pi\mathbf{P}\mathbf{W} + (1-\pi)\mathbf{V}$ where $\mathbf{V}$ is the vector $(\dots, V(d-1), V(d), V(d+1), \dots)$ and each element $(d, d+x)$ of $\mathbf{P}$ is given by:

$$P_{d,d+x} = \begin{cases} \tilde{p}_x(d) & \text{for } x = -1, 0, 1; \\ 0 & \text{otherwise.} \end{cases} \tag{A8}$$

Therefore $\mathbf{P}$ is a stochastic matrix with maximum eigenvalue of 1. Being $\pi < 1$, $(\mathbf{I}{-}\pi\mathbf{P})$ is invertible and hence obtain $\mathbf{W} = (1-\pi)\,(\mathbf{I}{-}\pi\mathbf{P})^{-1}\,\mathbf{V}$. By Caley-Hamilton theorem we can write

$$\mathbf{W} = (1-\pi)\sum_{n=0}^{\infty}\pi^n\mathbf{P}^n\mathbf{V},$$

showing that $W(d)$ is a convex combination of values between -1 and 1.

We can now prove that $W(1) > W(0) > W(-1)$. Consider the recursive relation given $\mathbf{V}^{(n)} = \mathbf{P}\mathbf{V}^{(n-1)}$, $n = 1, 2, \dots$ where $\mathbf{P}$ is defined as in (A8). Then $\mathbf{W} = (1-\pi)\sum_{n=0}^{\infty}\pi^n\mathbf{V}^{(n)}$, where $\mathbf{V}^{(0)} = \mathbf{V}$. We show by induction that for all n, $V^{(n)}(1) > V^{(n)}(0) > V^{(n)}(-1)$. Suppose this is true for a given n, then $V^{(n+1)}(d) = \sum_{x=-1}^{1}\tilde{p}_x(d)V^{(n)}(d+x)$ and hence $V^{(n+1)}(1) > V^{(n)}(1) > V^{(n+1)}(0) > V^{(n)}(-1) > V^{(n+1)}(-1)$. But this must be the case for $n = 0$ as $V(1) > V(0) > V(-1)$ hence $W(1) > W(0) > W(-1)$.[40]

Step 3. We are now in a position to characterize the solution to the system (13) and (14) for $|d| \geq 1$. Consider the problem in (13). Given player B's action $\tilde{b}$, this can be rewritten as:

$$W(d) - (1-\pi_t)V(d) = \pi_t \max_a\{\sum_{x=-1}^{1} p_x(a,\tilde{b})W(d+x)\}.$$

[40] It is easy to extend the proof to show strict monotonicity of $W(d)$ for all d.

Rearranging terms obtain:

$$(1-\pi)(W(d)-V(d)) = \pi \max_a \{p_1(a,\tilde{b})[W(d+1)-W(d)] - p_{-1}(a,\tilde{b})[W(d)-W(d-1)]\}.$$

The right hand side represents the difference between the expected marginal value of scoring a successful action and the expected marginal cost of conceding one. Notice that $W(d) < 1$ by Step 2.1 and $V(d) = 1$ for $d \geq 1$ then the marginal gain of a successful action is always less than its marginal costs. It follows that the equilibrium actions are given by $(\tilde{a}(d), \tilde{b}) = (0, 1)$. The opposite holds for $d \leq -1$ with equilibrium actions $(\tilde{a}(d), \tilde{b}(d)) = (1, 0)$.

Step 4: By following the steps in the proof of Case 1 in Lemma 1, it is possible to show that for $d = 0$ the solution must be in the interior and

$$(\tilde{a}(0), \tilde{b}(0)) = \arg\max_a \min_b \{p_1(a,b)[W(1)-W(0)] - p_{-1}(a,b))[W(0)-W(-1)]\}. \tag{A9}$$

We finally prove that the interior solution for the unknown duration game is the same as in the known duration one at $T-1$. First notice that $W(1) - W(0) = W(0) - W(-1)$. Suppose not and let $[W(1) - W(0)] < [W(0) - W(-1)]$. Then $(\tilde{a}(0), \tilde{b}(0)) = (0, 1)$ that is contradiction of the interior solution. A similar contradiction follows from $[W(1) - W(0)] > [W(0) - W(-1)]$ as in that case $(\tilde{a}(0), \tilde{b}(0)) = (1, 0)$. Therefore equation (A9) becomes:

$$(\tilde{a}(0), \tilde{b}(0)) = \arg\max_a \min_b \{p_1(a,b) - p_{-1}(a,b))\}. \tag{A10}$$

Now consider the equilibrium in the fixed duration game at $t = T-1$ and $d = 0$. Replacing the values for $V(1) = 1$, $V(0) = 0$ and $V(-1) = -1$ obtain:

$$(a^*(0), b^*(0)) = \arg\max_a \min_b \{p_1(a,b) - p_{-1}(a,b))\},$$

the same as in (A10). □

Lemma A1. *Player B's equilibrium action is given by* $\beta(a^*) = [f']^{-1}\left[C_b[1 - C_a^{-1} f'(a^*)]\right]$.

Proof of Lemma A1: From the equilibrium condition on the relative elasticities it is easy to show that $\epsilon^{A*} = [\epsilon^{B*}]^{-1}$. For the functional forms specified in (19) and (20) obtain:

$$\frac{C_a - f^{'}(a^*)}{f^{'}(a^*)} = \frac{f^{'}(\beta(a^*))}{C_b - f^{'}(\beta(a^*))} \;\Rightarrow\; f^{'}(\beta(a^*)) = \frac{C_b}{C_a}\left[C_a - f^{'}(a^*))\right], \tag{A11}$$

and hence the result. □

Lemma A2. *There is a unique turning point a_p such that $p_1(a^*)$ is locally concave (convex) at a_p if and only if $f''' > 0$ ($f''' < 0$).*

Proof of Lemma A2: Changes in $p_1(a^*, \beta(a^*))$ with respect to action a^* are obtained by solving:

$$\frac{dp_1(a^*)}{da^*} = \partial_a p_1(a^*, \beta(a^*)) + \partial_b p_1(a^*, \beta(a^*))\beta'(a^*), \tag{A12}$$

where $\partial_a p_1(a,b) = (C_a - f'(a))p_1(a,b) > 0$ and $\partial_b p_1(a,b) = f'(b)p_1(a,b) > 0$. Imposing the equilibrium condition in (A11) it is straightforward to show that:

$$\frac{dp_1(a^*)}{da^*} = \left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*), \tag{A13}$$

where $p_1(a^*, \beta(a^*)) = p_1(a^*)$. Note that there exists a turning point a_p of $p_1(a_p)$ such that $\frac{dp_1(a^*)}{da^*} = 0$ if and only if $\beta^{'}(a^*) = -\frac{C_a}{C_b}$. The first derivative of $\beta(a^*)$ can also be computed from (A11) obtaining: $\beta^{'}(a^*) = -\frac{C_b}{C_a}\frac{f^{''}(a^*)}{f''(\beta(a^*))}$. Equating the two values gives:

$$\left[\frac{C_b}{C_a}\right]^2 \frac{f^{''}(a_p)}{f''(\beta(a_p))} = 1, \tag{A14}$$

where $\beta(a_p)$ can be obtained by Lemma A1. Since $f^{''}$ is strictly monotonic the turning point a_p of $p_1(a_p)$ is unique.

Let us now show that the function $p_1(a^*)$ is locally concave at a_p. Taking the second derivative of $p_1(a^*, \beta(a^*))$ obtain:

$$\begin{aligned}
\frac{d^2p_1(a^*)}{d^2a^*} \equiv \frac{d^2p_1(a^*, \beta(a^*))}{d^2a^*} &= d\left[\left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*)\right] \\
&= -f^{''}(a^*)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*) + \frac{C_b}{C_a}\beta^{''}(a^*)\left(C_a - f^{'}(a^*)\right)p_1(a^*) \\
&+ \left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)\frac{dp_1(a^*)}{da^*} \\
&= \left[-f^{''}(a^*)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right) + \frac{C_b}{C_a}\beta^{''}(a^*)\left(C_a - f^{'}(a^*)\right)\right]p_1(a^*)
\end{aligned}$$

$$+\left(C_a - f^{'}(a^*)\right)^2 \left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)^2 p_1(a^*).$$

By equation (A14) at $a^* = a_p$ it follows that:

$$d^2 p_1(a_p) = \beta^{''}(a_p)\left(C_a - f^{'}(a_p)\right) p_1^*(a).$$

The latter is negative if and only if $\beta^{''}(a_p) < 0$. Computing $\beta^{''}(a_p)$ obtain:

$$\beta^{''}(a_p) = -\frac{C_b}{C_a}\frac{f^{'''}(a_p)f^{''}(\beta(a_p)) - \beta^{'}(a_p)f^{'''}(\beta(a_p))f^{''}(a)}{\left[f^{''}(\beta(a_p))\right]^2}.$$

Then $\beta^{''}(a_p) < 0$ if and only $f^{'''} > 0$ that is the necessary and sufficient condition for the local concavity of the equilibrium $p_1(a^*)$ at a_p. □

Proof of Proposition 4: For both type of games, the point d_p is the projection of a_p on d. Given t this be computed by using (15). □

Lemma A3. *There exists a unique action* $a_+ = [f^{'}]^{-1}(C_a/2)$ *and* $b_- = [f^{'}]^{-1}(C_b/2)$, *such that* $\mathcal{A}_+(b^*) = \{a : a < a_+\}$ *and* $\mathcal{B}_-(a^*) = \{b : b < b_-\}$.

Proof of Lemma A3: From (19) and (20) obtain: $\epsilon^A(a, b^*) = \frac{C_a - f^{'}(a)}{f^{'}(a)}$. Therefore, $\mathcal{A}_+(b^*) = \left\{a : \epsilon^A(a, b^*) > 1\right\} = \{a : \frac{C_a - f^{'}(a)}{f^{'}(a)} > 1\}$. Note that a_+ is unique since $\frac{\partial \epsilon^A(a,b^*)}{\partial a} = -\frac{C_a f^{''}(a)}{[f^{'}(a)]^2} < 0$ for all a. The proof for $\mathcal{B}_-(a^*)$ is similar. □

Proof of Proposition 5: We first show that $a_p \in \mathcal{A}_+^*$ if and only if $C_b > C_a$. Notice that, at equilibrium, from (A14) $\left[\frac{C_b}{C_a}\right]^2 f^{''}(a_p) = f^{''}(\beta(a_p))$. If $\frac{C_b}{C_a} < 1$ then $f^{''}(a_p) > f^{''}(\beta(a_p))$ and since $f^{''} > 0$ it follows that $a_p > \beta(a_p)$.

Using (A11) obtain: $f^{'}(\beta(a_p)) - f^{'}(a_p) = \frac{C_b}{C_a}\left[C_a - f^{'}(a_p)\right] - f^{'}(a_p)$. Being $f^{''} > 0$:

$$\begin{aligned}
&0 < f^{'}(\beta(a_p)) - f^{'}(a_p) = \frac{C_b}{C_a}\left[C_a - f^{'}(a_p)\right] - f^{'}(a_p) \\
&0 < \frac{C_b}{C_a}\left[C_a - f^{'}(a_p)\right] - \frac{C_b}{C_a}f^{'}(a_p) < \frac{C_b}{C_a}[C_a - 2f^{'}(a_p)] < [C_a - 2f^{'}(a_p)] \\
&\Rightarrow f^{'}(a_p) < \frac{C_a}{2} = f^{'}(a_+).
\end{aligned}$$

Thus $a_+ > a_p$ and hence $a_p \in \mathcal{A}_+^*$. From Lemma A2, if $f^{'''} > 0$ then $\frac{dp_1(a^*)}{da^*} < 0$ for $a < a_p$ ($\frac{dp_1(a^*)}{da^*} > 0$ for $a > a_p$). Since $\frac{da(t,d)}{dt} > 0$ for $a \in \mathcal{A}_+^*$ it follows that $\frac{dp_1(a^*(t,.))}{dt} < 0$ if $a^*(t,.) < a_p = a^*(t_p,.)$ (and $\frac{dp_1(a^*(t,.))}{dt} > 0$ if $a^*(t,.) > a_p = a^*(t_p,.)$), where t_p is a projection of a_p given d that can be computed using (16). On the complementary set $\mathcal{A}_-^*$ the function is monotonically decreasing overtime as $\frac{dp_1(a^*)}{da} > 0$ but $\frac{da(t,d)}{dt} < 0$. A similar argument proves that if $C_b < C_a$ then $a_p \in \mathcal{A}_-^*$. $\square$

Appendix B

Season	#Obs	Tot goals	RT goals	AT goals	Avg total	Avg RT	Avg AT
1995-96	1319	3597	3490	107	2.727	2.646	0.081
1996-97	1618	4366	4173	190	2.698	2.579	0.117
1997-98	1349	3636	3502	134	2.695	2.596	0.099
1998-99	2402	6126	5835	291	2.550	2.429	0.121
Pre-1998	**4286**	**11599**	**11165**	**431**	**2.706**	**2.605**	**0.101**
1999-00	2427	6451	6164	287	2.658	2.540	0.118
2000-01	1597	4415	4200	215	2.765	2.630	0.135
2001-02	1588	4228	4027	201	2.662	2.536	0.127
2002-03	1594	4263	4025	238	2.674	2.525	0.149
2003-04	1359	3684	3510	174	2.711	2.583	0.128
Post-1998	**15253**	**40766**	**38926**	**1837**	**2.673**	**2.552**	**0.128**
Total All	**19539**	**52365**	**50091**	**2268**	**2.680**	**2.564**	**0.116**

Table B1: Descriptive statistics