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**Resale in Second-Price Auctions with Costly
Participation**

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Resale in Second-Price Auctions with Costly Participation*

Gorkem Celik and Okan Yilankaya[†]

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Abstract

We study sealed-bid second-price auctions with costly participation and resale. Each bidder chooses to participate in the auction if her valuation is higher than her optimally chosen participation cutoff. If resale is not allowed and the bidder valuations are drawn from a strictly convex distribution function, the symmetric equilibrium (where all bidders use the same cutoff) is less efficient than a class of two-cutoff asymmetric equilibria. Existence of these equilibria without resale is sufficient for existence of similarly constructed two-cutoff equilibria with resale. Moreover, the equilibria with resale are “more asymmetric” and (under a sufficient condition) more efficient than the corresponding equilibria without resale.

JEL Classification: C72, D44, D82

Keywords: Second-price auctions; resale; participation cost; endogenous entry; endogenous valuations

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1 Introduction

We study resale in an independent private values auction setting with costly participation, with a particular focus on efficiency. The seller uses a sealed-bid second-price auction. Bidders are ex-ante symmetric: Their (use) values are drawn from the same distribution function. After learning their private valuations, bidders simultaneously decide whether to participate in the auction or not. Bidders who choose to participate incur a common real resource cost.^{1,2}

In the absence of resale opportunities, there is a (unique) symmetric equilibrium of the second-price auction where each bidder bids her valuation iff it is larger than a *participation cutoff* that is common to all bidders. However, there may also be asymmetric equilibria with bidder-specific cutoffs.³ We first show that, when the valuations are distributed according to a strictly convex cumulative distribution function, there are asymmetric equilibria which are ex-ante more efficient than the symmetric equilibrium. Existence of asymmetric equilibria under strict convexity has been established by Tan and Yilankaya (2006): For any arbitrary partition of the bidders into two groups, there exists an equilibrium where the bidders within a group all use the same participation cutoff that is different from the other group's cutoff. We complement this finding by showing that these two-cutoff equilibria provide a higher expected social surplus than the symmetric equilibrium (Proposition 1). The relevance of this result extends beyond second-price auctions, since Stegeman (1996) shows that one of the equilibria of the second-price auction maximizes social surplus within the class of all incentive-compatible allocation rules satisfying the “no passive reassignment” property.⁴

The second-price auction allocates the object to the highest valuation bidder *among participants* in all equilibria where participating bidders bid their

¹Purchasing bid documents, registering or pre-qualifying for the auction, being at the auction site, arranging for financing ahead of time and preparing a bid (which is often a detailed plan with documentation, especially in government procurement) are all costly activities.

²This set-up was introduced by Samuelson (1985), and studied by, among others, Stegeman (1996), Campbell (1998), Tan and Yilankaya (2006, 2007) and Celik and Yilankaya (2009). Also see Green and Laffont (1984), where costs as well as valuations are private information.

³See Stegeman (1996) for an example and Tan and Yilankaya (2006) for necessary and sufficient conditions for existence of asymmetric equilibrium.

⁴An allocation rule satisfies “no passive reassignment” if the object is assigned only to bidders participating in the auction. In Stegeman's (1996) example of a second-price auction with an asymmetric equilibrium, the asymmetric equilibrium is more efficient than the symmetric one. Celik and Yilankaya (2009) provide a characterization result for efficient auctions.

values. Yet, when the equilibrium is asymmetric, there is a possibility that a non-participating bidder has a higher valuation than the winner of the auction. This allocative inefficiency implies that there are potential gains from further trade through resale. Hence we incorporate the possibility of resale (assumed to be costless) via an optimal auction maximizing the reseller’s revenue, and study its impact on equilibrium behavior and efficiency.

Suppose that there exists a two-cutoff asymmetric equilibrium of the second-price auction without resale, where one group has a low cutoff and the other group has a higher one. We show that there also exists an equilibrium that partitions bidders the same way when resale is allowed. This resale equilibrium is “more asymmetric” than the corresponding no-resale equilibrium: The low cutoff decreases and the high cutoff increases (Proposition 2). The prospect of reselling the good induces the low-cutoff bidders to enter even more aggressively and the possibility of buying the object in the resale phase makes the high-cutoff bidders even more hesitant to enter. Moreover, there is overbidding by low-cutoff bidder types who hope to resell: They bid their adjusted values (payoffs inclusive of the resale phase), which are higher than their use values.

Fixing participation and bidding behavior in the initial auction, resale enhances efficiency as the object is potentially transferred to a higher-value bidder. However, the possibility of resale may also affect the equilibrium cutoffs and bids. Nevertheless, provided that a sufficient condition is satisfied, allowing resale improves ex-ante efficiency: Whenever there is a two-cutoff asymmetric equilibrium without resale, the corresponding more asymmetric equilibrium with resale yields a higher social surplus (Proposition 3). The efficiency gains from resale are not solely the result of savings in participation costs. First, our sufficient condition is on the distribution of valuations and hence *independent* of the magnitude of the participation cost. Second, allowing resale may actually increase participation, and hence total participation cost incurred. Similarly, this efficiency result is not an artifact of the modeling choice that there are no participation costs in the resale stage: We provide examples with costly bidding in resale where asymmetric equilibria exist and yield higher surplus than the corresponding no-resale equilibria. We discuss this issue further in our concluding remarks.

Resale is commonly observed after auctions in many markets. There are a few sources of gains from resale trade offered in the literature. New bidders or more information to existing bidders may arrive between the initial auction and the resale stage (Bikhchandani and Huang, 1989; Haile, 1996 and 2003; Bose and Deltas, 2006). Bidder asymmetries may also cause inefficiencies in first-price auctions (Gupta and Lebrun, 1999; Hafalir and Krishna, 2008; Cheng and Tan, 2010; Lebrun, 2010a; Virág, 2013) and in optimal auctions

maximizing the seller's revenue (Zheng, 2002).⁵

In second-price and English auctions, even when resale is allowed, (use) value-bidding remains to be an equilibrium (see, for example, Haile, 1996). This equilibrium allocates the object to the highest value bidder and hence there is no resale on the path of play. However, bidding one's value is no longer weakly dominant when resale is allowed. Garratt and Tröger (2006) identify alternative equilibria where even a speculator with no use value can make positive profits. Garratt, Tröger, and Zheng (2009) construct equilibria for the English auction with a designated bidder (potential reseller) which can then be used to support collusion (by rotating the designated bidder). In these equilibria, bidders other than the designated one drop out of the auction immediately, if their values are below a common cutoff.⁶ Our equilibrium construction for second-price auctions with costly participation follows a similar participation cutoff structure. However, unlike in Garratt, Tröger, and Zheng (2009), our asymmetric equilibria allow for designating *multiple* bidders who use a lower participation cutoff than the others and hence who all have the potential to resell the good. Regardless of whether they are low or high-cutoff bidders, all participants bid their adjusted values that reflect potential gains from resale. As we discussed above, when there are participation costs, second-price auctions (without resale) may have asymmetric equilibria in undominated strategies, where all participants bid their values. We investigate the resale opportunities naturally arising from these equilibria.

When resale takes place under asymmetric information, any equilibrium with a positive probability of resale would allocate the good in an ex-post inefficient manner: The bidder who ends up with the good at the end of the resale stage is not necessarily the bidder who values it the most. This is the reason that the equilibria identified by Garratt, Tröger, and Zheng (2009) induce a lower social surplus than the use-value bidding equilibrium. Such an allocative inefficiency is also present for the asymmetric equilibria of the second-price auction. Despite this allocative inefficiency, our paper introduces a welfare-enhancing role for resale when participation is costly.

The closest paper to ours is by Xu, Levin, and Ye (2013), who study second-price auctions with resale, where valuations and participation costs

⁵Also see Lebrun (2012).

⁶Similarly, Pagnozzi (2007) shows that a strong bidder may drop out of an ascending-price auction before the price reaches her value to improve her bargaining position in resale. Lebrun (2012) considers the second price auction with resale where two asymmetric bidders face a common reserve price and personalized entry fees. He shows that this auction has an equilibrium in mixed bidding strategies which yields the same revenue as in Myerson's (1981) optimal auction.

are both private information. They show that a symmetric equilibrium exists and is unique under some conditions. Participants in the initial auction bid their adjusted values and there is resale in equilibrium. Resale opportunities arise because of differences in participation costs: When a low-cost bidder wins the object, she can resell it to a high-cost bidder with a higher valuation (who did not participate in the initial auction). Further analytical results are difficult to obtain with two-dimensional private information. Their numerical analysis suggests that the effect of resale on efficiency (and on revenue) is ambiguous. In our model with commonly known participation costs, we show that heterogeneity of costs is not necessary for equilibrium resale. Instead, resale opportunities are generated by asymmetric equilibria. This setting also allows us to obtain an analytical result on the impact of resale on efficiency.

In the next section, we describe the environment and study the benchmark case, where resale is not allowed. We study resale in Section 3, analyzing the optimal resale auction, and the participation and bidding behavior in the initial auction. We provide some concluding remarks in Section 4. All proofs are in the Appendix.

2 No Resale

We consider a symmetric independent private values environment. There is a risk-neutral seller who owns an indivisible object and is selling it via a sealed-bid second-price auction without a reserve price. Her valuation is normalized to be 0. There are $n \geq 2$ risk-neutral (potential) bidders. Let v_i denote the (use) value of bidder $i \in \{1, \dots, n\}$ for the object. Bidders' valuations are independently distributed according to the cumulative distribution function (cdf) F on $[0, 1]$, with continuously differentiable and positive density function f . Bidders know their own valuations. We assume that the monotone hazard rate condition is satisfied: $\frac{1-F(v)}{f(v)}$ is strictly decreasing in v . Note that this condition automatically holds for convex distribution functions (for weakly increasing density functions f).

There is a participation cost, common to all bidders, denoted by $c \in (0, 1)$: Bidders must incur this real resource cost in order to be able to submit a bid.⁷ All bidders make their participation and bidding decisions simultaneously. They know their valuations when making these decisions.

⁷For our positive results about equilibria in auctions with or without resale, c can also be interpreted as an entry fee (charged by the seller).

2.1 Equilibrium

We first study the benchmark case where there is no resale possibility. If a bidder decides to participate in the second-price auction, she cannot do better than bidding her own valuation. Accordingly, we only consider (Bayesian-Nash) equilibria in *cutoff strategies*: Each bidder bids her valuation if it is greater than a cutoff point and does not participate otherwise.⁸ Even though all participating bidders bid in the same truthful manner, there may be asymmetric equilibria where bidders have different participation cutoffs. In what follows, we restrict attention to equilibria with (at most) two cutoffs. Since our results are of the existence/possibility variety, this restriction has no bearing on them, while simplifying the exposition considerably.⁹

Suppose that l bidders use the low cutoff a and $h = n - l$ bidders use the high cutoff b in some equilibrium, with $0 \leq a \leq b \leq 1$ and $1 \leq l \leq n$. These cutoffs are determined by indifference (to participation) conditions. To find them, first consider the participation decision of one of the l bidders who has the lower cutoff a and whose valuation is also a . Suppose that all other bidders are following their equilibrium strategies. She obtains the object iff she is the only bidder to participate, which happens with probability $F(a)^{l-1} F(b)^h$, and hence pays 0 if she wins. Her expected payoff from participation is then

$$F(a)^{l-1} F(b)^h a - c. \quad (1)$$

Similarly, the expected payoff of a high-cutoff bidder with valuation b is

$$\begin{aligned} & F(a)^l F(b)^{h-1} b + F(b)^{h-1} \int_a^b (b-w) dF(w)^l - c \\ = & F(b)^{h-1} [F(a)^l a + \int_a^b F(w)^l dw] - c. \end{aligned} \quad (2)$$

Define the following functions: $\tilde{\pi}_L(a, b) = F(a)^{l-1} F(b)^h a$ and $\tilde{\pi}_H(a, b) = F(b)^{h-1} [F(a)^l a + \int_a^b F(w)^l dw]$.

The following conditions are necessary and sufficient for (a^*, b^*) to be equilibrium cutoffs:

$$\begin{aligned} \tilde{\pi}_L(a^*, b^*) & \geq c, \text{ with equality if } a^* > 0, \\ \tilde{\pi}_H(a^*, b^*) & \leq c, \text{ with equality if } b^* < 1. \end{aligned} \quad (3)$$

⁸We adopt the convention that if a bidder's cutoff is 0 (respectively, 1) all (respectively, none of) her types are participating. The participation decisions of these zero measure cutoff types are inconsequential.

⁹Tan and Yilankaya (2006) show that, for *any* c , log-concavity of $F(\cdot)$ is sufficient for the upper limit of distinct cutoffs to be two in any (cutoff) equilibria.

Any bidder with a value lower than c will have a negative payoff from participation. So, we know that $a^* \geq c > 0$, and the first condition will be satisfied with equality.¹⁰

We note the following observation, which will be used later:

Remark 1 $\tilde{\pi}_L(a, b)$ and $\tilde{\pi}_H(a, b)$ are strictly increasing in a and b for $a > 0$. ($\tilde{\pi}_L(0, b) = 0 \forall b$ and $\tilde{\pi}_H(0, b)$ is strictly increasing in b .)

Notice that, since we allow for the possibility that $a = b$, the symmetric equilibrium is a special case within the class of (at most) two cutoff equilibria. There always exists a symmetric equilibrium where all bidders use the same participation cutoff $a = b = v_s \in (c, 1)$, where

$$F(v_s)^{n-1} v_s = c. \quad (4)$$

Tan and Yilankaya (2006) show that strict convexity of F is sufficient for existence of two-cutoff asymmetric equilibria for *any* l , the number of bidders using the lower cutoff. It may be helpful to go over their argument with graphs, which we will also utilize for our results. Consider the set $\Delta = \{(a, b) : 0 \leq a \leq b \leq 1\}$ in R^2 that identifies feasible participation cutoff pairs, and the curves given by $\tilde{\pi}_L(a, b) = c$ and $\tilde{\pi}_H(a, b) = c$ in Δ . When F is strictly convex, the second curve is steeper than the first one at (v_s, v_s) , the symmetric equilibrium cutoffs. If these curves intersect in the interior of Δ , their intersection yields the cutoffs for an asymmetric equilibrium, satisfying (3) with equalities (Figure 1). Otherwise, we have a corner asymmetric equilibrium with $a^* \in [c, v_s)$ and $b^* = 1$, where $\tilde{\pi}_L(a^*, 1) = c$ and $\tilde{\pi}_H(a^*, 1) \leq c$, as depicted in Figure 2.

2.2 Efficiency

We can write down the social surplus as a function of the two cutoff points in the no-resale setting:

$$\tilde{S}(a, b) = \int_a^b vF(b)^h dF(v)^l + \int_b^1 v dF(v)^{h+l} - l(1 - F(a))c - h(1 - F(b))c. \quad (5)$$

The first integral measures the expected value of the object for the winner of the auction when she is a low cutoff bidder with a value on interval $[a, b]$, and

¹⁰We keep the conditions in their current forms to make them directly comparable to the corresponding conditions in the resale case, where a bidder with valuation less than c may participate in order to sell later.

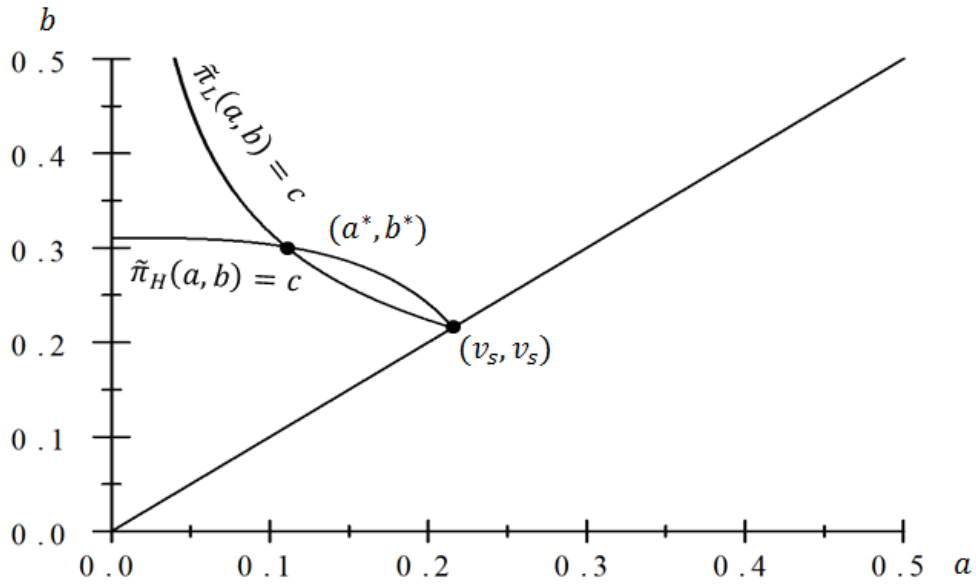


Figure 1: $F(v) = v^2, n = 2, c = 0.01$. Symmetric equilibrium with $v_s = 0.215$. Asymmetric equilibrium with $a^* = 0.11, b^* = 0.301$.

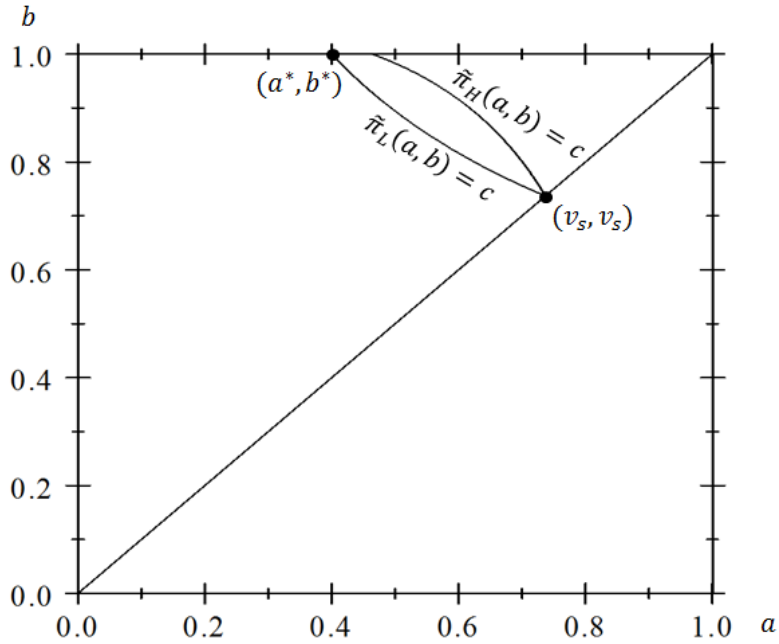


Figure 2: $F(v) = v^2, n = 2, c = 0.4$. Symmetric equilibrium with $v_s = 0.737$. Asymmetric equilibrium with $a^* = 0.4, b^* = 1$.

the second one is the expected value for a winner with a valuation higher than b . The last two terms are expected participation costs incurred by all bidders. Note that the seller's valuation is normalized to be 0 and the payment made by the winning bidder is just a transfer to the seller.

The derivatives of this social surplus function with respect to its two arguments can be written by referring to functions $\tilde{\pi}_L$ and $\tilde{\pi}_H$ that we just defined:

$$\begin{aligned}\frac{\partial \tilde{S}(a, b)}{\partial a} &= -lf(a) [\tilde{\pi}_L(a, b) - c], \\ \frac{\partial \tilde{S}(a, b)}{\partial b} &= -hf(b) [\tilde{\pi}_H(a, b) - c].\end{aligned}\tag{6}$$

Therefore, the social surplus is decreasing in a and increasing in b for the set of points where $\tilde{\pi}_H(a, b) \leq c \leq \tilde{\pi}_L(a, b)$, i.e., the lens-shaped areas in Figures 1 and 2. Accordingly, when F is convex, social surplus will be higher on the asymmetric equilibria identified by Tan and Yilankaya (2006) in comparison to the symmetric equilibrium.¹¹

Proposition 1 *If F is strictly convex, then, for any $l \in \{1, 2, \dots, n - 1\}$, there exists an asymmetric no-resale equilibrium, where l bidders use cutoff a^* and $h = n - l$ bidders use cutoff $b^* > a^*$, that generates a higher social surplus than the symmetric equilibrium.*

This result holds regardless of the magnitude of the participation cost c , the number of bidders n , and the way bidders are classified into low and high-cutoff groups. For given levels of c and n , strict convexity of F in Proposition 1 can be replaced with the weaker local condition that $\frac{F(v)}{v}$ is strictly increasing at the symmetric cutoff v_s , defined in (4). This local condition is all that is needed to generate the lens-shaped areas such as those in Figures 1 and 2.

¹¹In both of the examples in Figures 1 and 2, the asymmetric equilibrium payoff of the low-cutoff bidder is (weakly) higher than her symmetric equilibrium payoff (regardless of her valuation). The payoff ranking of the equilibria is reversed for the high-cutoff bidder. These equilibrium properties are most evident in the second example, since the high-cutoff bidder never participates under the asymmetric equilibrium and the low-cutoff bidder acquires the good by incurring the participation cost only. But for both examples, the sum of the ex-ante expected payoffs of the two bidders is larger under the asymmetric equilibrium. Moreover, if the role of the low-cutoff bidder is assigned to the two bidders with equal probabilities, then both bidders are weakly better off at the interim stage under the asymmetric equilibrium, regardless of their valuations. This equal assignment procedure can be supported if the bidders have access to a public randomization device, as in Garratt, Tröger, and Zheng (2009), or if they are facing each other repeatedly in a series of independent auctions, as in Bikhchandani and Riley (1991).

Strict convexity implies that $\frac{F(v)}{v}$ is strictly increasing for all v .¹² Finally, note that the result identifies at least $n - 1$ distinct (ignoring permutations of bidder identities) asymmetric equilibria of the second-price auction, and each of them has a higher surplus than the symmetric equilibrium.¹³

3 Resale

Asymmetric equilibria of the second-price auction with participation costs, such as those we discussed above, have the following feature: Even though the object is obtained by the bidder who has the highest valuation *among participants*, a nonparticipant may have a higher valuation. Therefore, when the winner of the auction has a valuation which is lower than the participation cutoff of another bidder, there are potential gains from further trade. To investigate this issue, we now allow for a resale stage where the winner of the initial auction can design her own resale auction for potential bidders.

Timing is as follows: Bidders make participation and bidding decisions simultaneously in the initial auction. The winner designs a resale auction if she chooses to do so. Others make their simultaneous participation and bidding decisions in this resale auction.

We assume that the highest bidder learns that she is the winner and does not learn others' bids.¹⁴ We also assume that there are no participation costs in the resale stage, and discuss this assumption in our concluding remarks.

¹²Since the second-price auction has an (ex-ante) efficient equilibrium in this setting (Stegeman, 1996), Proposition 1 implies that the efficient auction is asymmetric for strictly convex distribution functions. There is a connection between optimal (revenue-maximizing) and efficient auctions. Following Myerson (1981), define $J(v) = v - \frac{1-F(v)}{f(v)}$ as the *virtual value* of a bidder with value v . In Celik and Yilankaya (2009), we showed that, if $\frac{F(v)}{J(v)}$ is strictly increasing, then the optimal auction is asymmetric, implying (using the connection we mentioned) that the efficient auction is asymmetric when $\frac{F(v)}{v}$ is strictly increasing.

¹³The equilibria identified in this proposition are the only asymmetric equilibria when $F(\cdot)$ is log-concave (see Footnote 9). Using the results in Celik and Yilankaya (2009) and the parallels between the efficient and the optimal auctions, we can show that the social surplus is maximized with an asymmetric equilibrium where $l = 1$ and $h = n - 1$ under log-concavity.

¹⁴We make this no-disclosure assumption only for notational simplicity. Full disclosure of all bids (or any other intermediate disclosure policy) would not affect our equilibrium outcome since our equilibrium construction is based on adjusted value-bidding: The participants in the initial auction will bid their adjusted values inclusive of the expected resale payoff and no bidder in the initial auction will participate as a buyer in the resale phase. See Lebrun (2010b) for a discussion of importance of bid disclosure policies, especially in first-price auctions with resale.

We look for (Perfect Bayesian) equilibria of this game where bidders are divided into two groups that use two (potentially distinct) participation cutoffs in the initial auction, just like before. Similarly, we restrict attention to equilibria in which participants in the initial auction bid their adjusted values (gross expected payoff inclusive of the resale stage).¹⁵

We analyze the optimal resale auction first (given the restrictions above), followed by bids and equilibrium participation cutoffs in the initial auction.

3.1 Optimal resale auction

Suppose that l bidders use cutoff a and $h = n - l$ bidders use cutoff b in the initial auction, with $0 \leq a \leq b \leq 1$ and $1 \leq l \leq n$. Suppose further that bids are monotone increasing in valuations and that bidders with identical valuations bid the same amount (if they participate). In these equilibria we are constructing, there are opportunities for resale only if a bidder wins the initial auction with a value between a and b . The bidders who are using the higher cutoff b in the initial auction are the potential buyers in the resale stage. The winner of the initial auction (one of the l bidders who use a as the cutoff and who has a valuation in $[a, b]$) has learned that none of these high-cutoff bidders have a value higher than b , otherwise they would have participated in the initial auction and acquired the good. Therefore, the problem she is facing is finding the optimal auction for $h \geq 1$ bidders whose valuations are independently distributed on $[0, b]$ according to the cdf $\frac{F(\cdot)}{F(b)}$.^{16,17}

¹⁵Note that bidding this “adjusted value” is no longer the dominant strategy even conditional on participating, since this value is calculated using the equilibrium expected payoff from the resale auction. As we discussed before, Garratt and Tröger (2006), Garratt, Tröger, and Zheng (2009), and Lebrun (2012) identify other equilibria of second-price auction with resale, where bidders do not bid these adjusted values. Existence of such alternative equilibria is not pertinent to our results, which are of the existence/possibility variety.

¹⁶Since cdf $F(\cdot)$ satisfies the monotone hazard rate condition, so does the conditional cdf $F(\cdot)/F(b)$ (e.g., Hafalir and Krishna, 2008 and Cheng and Tan, 2010). This condition ensures that we are in the “regular case” with increasing virtual valuations. Notice also that when $h = 0$ (or $a = b$), we have symmetric participation in the initial auction, and hence there is no room for resale.

¹⁷We are describing the resale stage only on the equilibrium path (or rather when the initial auction behavior is described by two participation cutoffs and monotone bid functions). We do not formally define the full strategies at the resale stage to keep the exposition simple. Recall that the bids made in the initial auction are not disclosed. However, the equilibrium outcome we describe is robust to disclosure of bids: There are many resale-stage beliefs that would support this outcome including the passive one where the reseller does not update when she sees an off-the-equilibrium-path bid.

If the reseller’s valuation were commonly known, this would be the standard optimal auction problem à la Myerson (1981). However, it is not known, and so we have an “informed principal” problem.¹⁸ Fortunately, it is possible to show that this does not matter in this independent private values setting. It is optimal for each type of the reseller in $[a, b]$ to choose a standard optimal auction for that type.¹⁹

There are many auctions which are expected-payoff equivalent for the reseller and the bidders (the revenue equivalence theorem), but we will focus on a second-price auction with an optimal reserve price $r(w)$ for the reseller with valuation $w \in [a, b]$, satisfying^{20,21}

$$r = w + \frac{F(b) - F(r)}{f(r)}. \quad (7)$$

The monotone hazard rate condition implies that, for any b , the right hand side of (7) is decreasing in r for $r \in [0, b]$. Thus there is a unique value for $r(w) \in (w, b]$. Note that we have $r'(w) \in (0, 1)$ and $r(b) = b$.

A bidder with value v participates in the resale auction if $v \geq r$, and bids v . It is straightforward to calculate the expected payment she makes to the reseller if she wins the resale auction:²²

$$\alpha(v, r) = v - \int_r^v \frac{F(x)^{h-1}}{F(v)^{h-1}} dx. \quad (8)$$

¹⁸Garratt, Tröger, and Zheng (2009) avoid this problem by restricting attention to resale mechanisms that the reseller cannot participate by sending a message at the same time as (or after) the other bidders. They allow for bidder valuations to be drawn from different distributions. They highlight certain properties of Myerson’s optimal auction when it is used as a resale mechanism for these bidders. Our optimal resale auction is much simpler since the values of the potential bidders are identically distributed at the resale stage.

¹⁹Yilankaya (1999) shows this in the bargaining context, i.e., when $h = 1$. The same argument applies for $h > 1$ (Yilankaya, 2004, available from the authors upon request): The Myerson auction is optimal when the seller’s valuation is common knowledge. It is also the seller’s ex-ante optimal mechanism. Myerson’s *principle of inscrutability* (1983) implies that it will be the informed principal’s optimal mechanism. Also see the discussions in Maskin and Tirole (1990), Garratt, Tröger, and Zheng (2009), and Mylovanov and Tröger (2014).

²⁰On the equilibrium path, it does not matter which of the optimal auctions is used in the resale stage. However, the choice of the resale auction matters when considering potential deviations in the initial auction.

²¹We suppress the dependence of r on b for notational simplicity.

²²This expression may be familiar as the equilibrium bid function in a first-price auction with h bidders whose valuations are distributed according to $\frac{F(\cdot)}{F(b)}$ on $[0, b]$. Revenue equivalence theorem implies that this is the expected payment of the winner in the second-price auction.

3.2 Equilibrium bids in the initial auction

Now that we discussed the optimal resale auction on the equilibrium path, we are ready to study bidding in the initial auction. As we mentioned above, we look for an equilibrium in which bids are given by gross expected payoffs, taking the resale stage into account. When the winner is a bidder with a valuation higher than the high cutoff b , there is no room for resale and the winner's payoff is equal to her (use) value. On the other hand, when the winner is a low-cutoff bidder with a valuation $v \in [a, b]$, her gross payoff is equal to the expected continuation payoff in the resale stage. In the Appendix (Lemma 1), we show that this continuation payoff is $b - \int_v^b \frac{F(r(x))^h}{F(b)^h} dx$, where $r(x)$ is the optimal reserve price for a bidder with valuation x , by using the revenue equivalence theorem.²³

The next step is establishing the existence of an equilibrium where each participant in the initial auction bids her adjusted value. This result is not trivial, since each bidder's resale stage payoff also depends on her own bid in the initial auction. We show in the Appendix (Lemma 2) that, conditional on participating according to their respective cutoffs, the optimal bid for a low-cutoff bidder is

$$\beta(v) = \begin{cases} No & \text{if } v < a \\ b - \int_v^b \frac{F(r(x))^h}{F(b)^h} dx & \text{if } a \leq v < b \\ v & \text{if } b \leq v \end{cases}, \quad (9)$$

and the optimal bid for a high-cutoff bidder is

$$\hat{\beta}(v) = \begin{cases} No & \text{if } v < b \\ v & \text{if } b \leq v \end{cases}, \quad (10)$$

where “ No ” denotes not participating.

High-cutoff bidders bid their use values if they participate, since their participation precludes a resale stage. The equilibrium bid of a low-cutoff bidder is higher than her use value when it is in $[a, b]$. Bidders with such valuations are aware of the possibility that they can resell the good to a high-cutoff bidder who has a higher use value. Since this possibility is decreasing in the valuation of the low-cutoff bidder, the extent of overbidding is decreasing in v (and it is eliminated for $v \geq b$).

²³Garratt, Tröger, and Zheng (2009) use a similar argument to the proof of this Lemma to find the resale-stage payoffs of bidders in an English auction.

Bidding functions $\beta(\cdot)$ and $\hat{\beta}(\cdot)$ imply that the initial auction is ex-post efficient in a constrained sense: It allocates the good to the bidder with the highest valuation among participants. Any inefficiency in the initial auction (therefore, any resale opportunity) is due to possible asymmetric participation behavior, which we discuss next.

3.3 Equilibrium participation in the initial auction

A bidder's participation decision in the initial auction will depend on the comparison of the participation cost with the payoff differential generated by her participation, taking into account the resale stage. In the Appendix (Lemma 3), we show that, for each bidder, this payoff differential is (weakly) increasing in her valuation when other bidders are using cutoff strategies. Therefore, it is sufficient to consider participation incentives for bidders whose valuations are equal to their respective cutoffs.

Consider one of the l low-cutoff bidders who has a valuation equal to her cutoff a . This bidder cannot buy the object in a resale auction since all the equilibrium reserve prices will be higher than a . When she enters in the initial auction, she would be the winner if she is the only participant. Given the other bidders' participation decisions, the probability of this happening is $F(a)^{l-1} F(b)^h$. This sole participant does not make any payment, and her expected payoff in the continuation game is $\beta(a)$. Therefore, her payoff differential for participation in the initial auction is

$$\pi_L(a, b) = F(a)^{l-1} F(b)^h \beta(a). \quad (11)$$

Now consider one of the h high-cutoff bidders who has a valuation equal to her cutoff b . If she participates (and bids b), then her expected payoff will be

$$F(b)^{h-1} [F(a)^l b + \int_a^b [b - \beta(w)] dF(w)^l]. \quad (12)$$

To see this, notice that she wins only if none of the $h - 1$ high-cutoff bidders participates and all the l low-cutoff bidders have valuations less than b . She pays 0 if none of the low-cutoff bidders participates (when all of them have valuations less than a). Otherwise she pays the bid of the highest-valuation low-cutoff bidder, $\beta(w)$.²⁴

²⁴There will not be a resale stage if she participates, since resale happens only if a low-cutoff bidder wins the object with valuation in $[a, b]$, and these bidder types bid less than b , i.e., $\beta(w) < b$ for $w < b$.

On the other hand, if she stays out, then her expected payoff will be

$$F(b)^{h-1} \int_a^b [b - \alpha(b, r(w))] dF(w)^l, \quad (13)$$

since resale auction occurs if none of the $h-1$ other high-cutoff bidders participates in the initial auction and the highest valuation among low-cutoff bidders is in $[a, b]$. The bidder who has this highest valuation w sets the reserve price $r(w)$, so the expected payment of type- b bidder is $\alpha(b, r(w))$ (see (8)).

Therefore, the payoff differential for a high-cutoff bidder with valuation b is

$$\pi_H(a, b) = F(b)^{h-1} [F(a)^l b + \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l]. \quad (14)$$

We prove the following in the Appendix.

Remark 2 $\pi_L(a, b)$ is increasing in a and b for $a > 0$, and $\pi_H(a, b)$ is increasing in a .

We are finally ready to identify the conditions that equilibrium cutoffs a^{**} and b^{**} must satisfy, analogous to conditions (3) in the no-resale setting:

$$\begin{aligned} \pi_L(a^{**}, b^{**}) &\geq c, \text{ with equality if } a^{**} > 0, \\ \pi_H(a^{**}, b^{**}) &\leq c, \text{ with equality if } b^{**} < 1. \end{aligned} \quad (15)$$

These conditions admit a symmetric solution, with $a^{**} = b^{**} = v_s$, the symmetric equilibrium cutoff of the benchmark no-resale case. There is no resale in this equilibrium, since the bidder with the highest valuation receives the object.

Our next result is about the existence and properties of equilibria with asymmetric cutoffs, where resale is an equilibrium phenomenon. Whenever there is an asymmetric equilibrium (with two cutoffs) in the benchmark no-resale case, there will also be an asymmetric equilibrium with resale. Moreover, the equilibrium with resale will be “more asymmetric.”

Proposition 2 *Suppose that there exists an asymmetric equilibrium in the benchmark case of no-resale with $l \in \{1, 2, \dots, n-1\}$ bidders using the cutoff a^* and $h = n-l$ bidders using the cutoff $b^* > a^*$. Then there exists an asymmetric equilibrium with resale, where l bidders use cutoff a^{**} and h bidders use cutoff $b^{**} > a^{**}$. Moreover, $a^{**} < a^*$ and $b^{**} \geq b^*$ (with strict inequality if $b^* < 1$).*

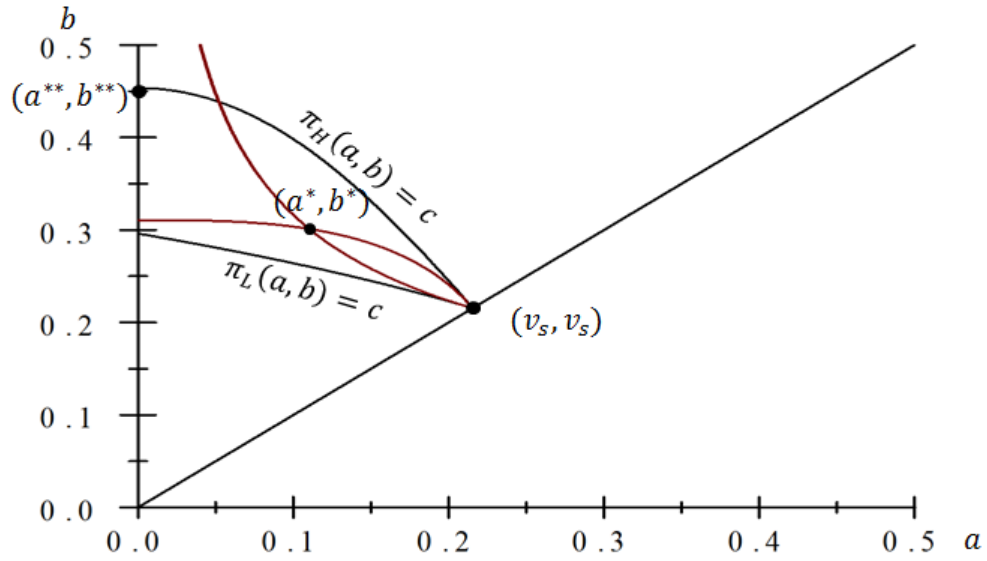


Figure 3: $F(v) = v^2, n = 2, c = 0.01$. Symmetric equilibrium with $v_s = 0.215$. Asymmetric equilibrium with $a^{**} = 0, b^{**} = 0.454$.

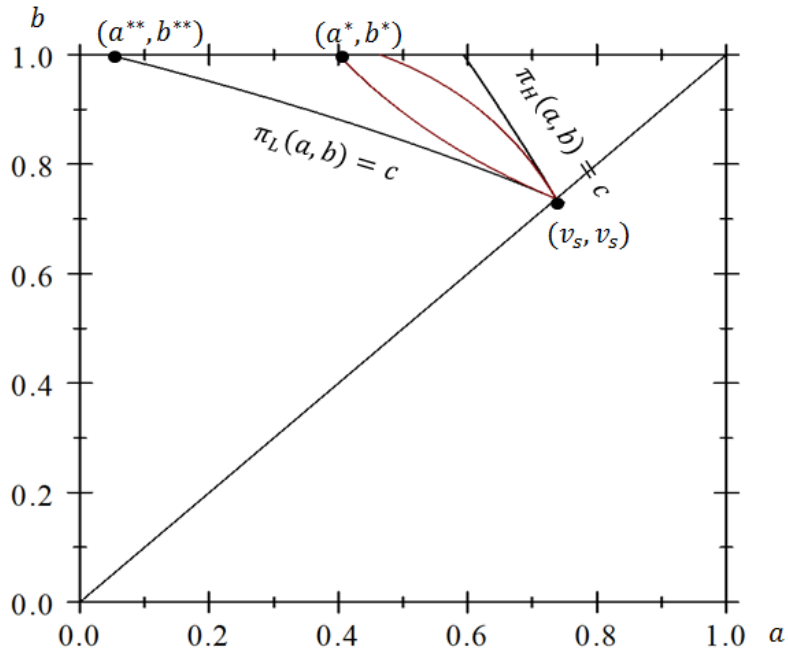


Figure 4: $F(v) = v^2, n = 2, c = 0.4$. Symmetric equilibrium with $v_s = 0.737$. Asymmetric equilibrium with $a^{**} = 0.044, b^{**} = 1$.

A key step in our proof is showing that $\pi_L(a, b) > \tilde{\pi}_L(a, b)$ and $\pi_H(a, b) < \tilde{\pi}_H(a, b)$ for $b > a > 0$. These inequalities imply that the asymmetric no-resale equilibrium cutoffs (a^*, b^*) lie below the curve given by $\pi_H(a, b) = c$ and above the curve $\pi_L(a, b) = c$. The remainder of the argument is very similar to our discussion of asymmetric equilibria in the no-resale benchmark. If these two curves intersect in the interior of $\Delta = \{(a, b) : 0 \leq a \leq b \leq 1\}$, their intersection yields the cutoffs for an asymmetric equilibrium with resale, satisfying (15) with equalities. Otherwise, we have a corner equilibrium either with $a^{**} = 0$ (Figure 3) or with $b^{**} = 1$ (Figure 4). The resulting equilibrium cutoffs (a^{**}, b^{**}) are more asymmetric than the corresponding no-resale equilibrium cutoffs (a^*, b^*) in the sense that they are further away from the symmetric equilibrium cutoff v_s .

In an equilibrium such as the one described above, a low-cutoff bidder participates and bids more aggressively in the initial auction due to the opportunity to resell to a high-cutoff bidder. This opportunity in turn is supported by some types of the high-cutoff bidder remaining out of the initial auction to buy later in the resale stage. This asymmetry in behavior arises as an equilibrium phenomenon even though bidders are ex-ante symmetric, as it is the case in the no-resale benchmark. A similar speculative motive also appears in the symmetric equilibrium of Xu, Levin and Ye's (2013) model with two possible (privately-known) participation costs: Bidders with high cost tend to stay out of the initial auction and the low-cost bidders over-enter and over-bid with the hope of reselling the object. With Proposition 2 we show that this speculative motive and resale can arise in equilibrium even when all bidders have the same participation cost.

3.4 Efficiency with resale

To examine the welfare effects of resale, we consider the social surplus as a function of two participation cutoffs. This surplus function is constructed under the assumption that, once the bidders enter in or stay out of the initial auction according to these participation cutoffs, they follow the equilibrium bidding and resale strategies described above.

$$S(a, b) = \int_a^b \left[F(r(w))^h w + \int_{r(w)}^b v dF(v)^h \right] dF(w)^l \quad (16)$$

$$+ \int_b^1 w dF(w)^{h+l} - l(1 - F(a))c - h(1 - F(b))c.$$

The first integral term refers to the expected surplus if the initial auction allocates the good to a low-cutoff bidder with valuation in $[a, b]$. This expected

surplus is calculated by taking the possibility of resale into account. The second integral term is the expected surplus when the highest valuation among all bidders is higher than cutoff b . The last two terms measure the expected cost of participation.

For fixed cutoffs, the possibility of resale increases total welfare: $S(a, b) > \tilde{S}(a, b)$ for $a < b$, since

$$S(a, b) - \tilde{S}(a, b) = \int_a^b \int_{r(w)}^b (v - w) dF(v)^h dF(w)^l. \quad (17)$$

This difference is simply the surplus gain of transferring the object from a low-cutoff bidder with value w to a high-cutoff bidder with a higher value v in the resale phase.

Consider an equilibrium in the benchmark case of no-resale with asymmetric cutoffs a^* and b^* . As we just observed, if bidders were to use the same cutoffs when resale is allowed, then the surplus would be higher, i.e., $S(a^*, b^*) > \tilde{S}(a^*, b^*)$. However, the possibility of resale may also change equilibrium participation behavior of the bidders. Therefore, we need to know how the value of function $S(a, b)$ changes as we move from the no-resale equilibrium cutoffs (a^*, b^*) to resale equilibrium cutoffs (a^{**}, b^{**}) . With our next result, we provide a sufficient condition for the social surplus to increase when resale is allowed.

Proposition 3 *Suppose that there exists a two-cutoff asymmetric equilibrium in the benchmark case of no-resale and that $\frac{vf(v)}{F(v)}$ is weakly increasing. Then there exists an asymmetric equilibrium with resale which generates a higher social surplus than does this asymmetric no-resale equilibrium.*

To prove the proposition, we first show that the social surplus function $S(a, b)$ is decreasing in a and increasing in b whenever $\pi_H(a, b) < c < \pi_L(a, b)$, if $\frac{vf(v)}{F(v)}$ is weakly increasing. The result then follows from inequalities $a^{**} < a^*$ and $b^{**} \geq b^*$.

Proposition 3 provides a sufficient condition on the distribution of valuations only. Hence, it is independent of the magnitude of the participation cost c , the number of bidders n , and how these bidders are divided into two groups. The condition is satisfied when the cdf for the valuations is a power function, i.e., $F(v) = v^\alpha$ for $\alpha > 0$ (since $\frac{vf(v)}{F(v)} = \alpha$).²⁵

²⁵For $F(v) = v^\alpha$ with $\alpha > 1$, it follows from our Proposition 1 together with the results of Stegeman (1996) and Tan and Yilankaya (2006) that one of the two-cutoff equilibria of the second-price auction (without resale) is efficient: This equilibrium maximizes social surplus

4 Concluding Remarks

We study resale and show that it can be an equilibrium phenomenon in a symmetric second-price auction with costly participation. The equilibrium with resale is more asymmetric than the corresponding one without resale due to speculative motives. When resale is not allowed, we identify asymmetric equilibria that are more efficient than the symmetric one if the cdf of bidders' valuations is strictly convex. We provide a sufficient condition for resale to improve (ex-ante) efficiency. Therefore, when F is strictly convex and this sufficient condition is satisfied, we have a ranking: Symmetric equilibrium (resale allowed or not) is less efficient than the asymmetric no-resale equilibria we identified, which in turn are less efficient than the corresponding asymmetric resale equilibria.

There is ambiguity for the effect of resale on the expected number of participants, and hence on the expected participation cost incurred. This can be seen by considering the examples depicted in Figures 3 and 4, where $n = 2$ and $F(v) = v^2$. When $c = 0.01$, no-resale asymmetric equilibrium cutoffs are $(a^*, b^*) = (0.11, 0.301)$, while with resale equilibrium cutoffs are $(a^{**}, b^{**}) = (0, 0.454)$, with a decrease in participation. However, resale increases participation when $c = 0.4$, because the equilibrium cutoffs change from $(0.4, 1)$ to $(0.044, 1)$: The high-cutoff bidder does not participate in either case while the low-cutoff bidder is more likely to participate when there is resale. This example also demonstrates that the efficiency gains from resale are not only due to savings in participation costs.

We assume that there are no participation costs at the resale stage. One possible justification is that the reseller may follow a bidder qualification procedure which is less stringent than that of the original seller, e.g., due to the original seller being a public entity (see Xu, Levin, and Ye, 2013, who also assume costless resale). However, the main reason for our assumption is to keep the analysis simple. When there are participation costs and more than one potential bidders for the resale (i.e., $h > 1$), the optimal resale auction would be more complicated than a standard auction; in particular, it could be asymmetric (Celik and Yilankaya, 2009). Nevertheless, we would like to stress that the welfare improvement under resale is not an artifact of the costless resale assumption. To illustrate this point, we reconsider the examples in

within the class of incentive-compatible allocation rules that can assign the object only to bidders participating in the initial auction. (See Footnotes 4, 9, and 12.) Our Proposition 3 implies that resale improves efficiency further by introducing the possibility of allocating the good to an initially non-participating bidder. The same observation applies to *maximum entropy* distribution functions with a mean larger than $1/2$, i.e., $F(v) = \frac{e^{\lambda v} - 1}{e^\lambda - 1}$ with $\lambda > 0$.

Figures 3 and 4 (with $n = 2$ and $F(v) = v^2$) under the alternative assumption that bidding in the resale auction is as costly as bidding in the initial auction. The new participation cutoffs are different than they were in the costless resale case: The asymmetric costly-resale equilibrium cutoffs are $(0, 0.433)$ for participation cost $c = 0.01$, and they are $(0.376, 1)$ for participation cost $c = 0.4$. For either cost level, the social surplus of the asymmetric costly-resale equilibrium (net of the participation costs in the initial auction and in the resale stage) is larger than the corresponding no-resale equilibrium social surplus.²⁶

The effect of resale on expected revenue is also unclear. Opportunity to resell the object induces higher participation and higher bids by some of the bidders, leading to a positive impact on revenue. Other bidders, however, would participate less since they might have the option to buy the object later, identifying a countervailing factor. Therefore, the net effect of resale on revenue is ambiguous, as can be seen in the following classes of examples: In the asymmetric no-resale equilibrium we construct, if some bidders never participate in the auction regardless of their values and at least two other bidders participate with positive probability (i.e., if $b^* = 1$ and $l > 1$), revenue would be higher in the corresponding equilibrium with resale (since the participating bidders will enter with higher probability and bid more). On the other hand, allowing for resale would eliminate all revenue if the asymmetric equilibrium with resale has only one bidder participating with a positive probability (i.e., if $b^{**} = 1$ and $l = 1$).

Finally, in our model, bidders make their participation and bidding decisions simultaneously. Another possibility is bidders making their participation decisions first and then bidding after having observed the number of participants (e.g., Xu, Levin, and Ye, 2013). This alternative scenario (or even observing the identities of participants) would not change our results, since in equilibrium all bidder types bid their adjusted values inclusive of the payoff from resale stage.

²⁶As one would expect, the costly-resale asymmetric equilibria produce a lower social surplus than their costless-resale counterparts.

5 Appendix

Proof of Proposition 1 (no-resale)

Following the proof of Proposition 3i in Tan and Yilankaya (2006), for all $b \in [v_s, 1]$, define $\Phi(b)$ implicitly as the value of a which solves $\tilde{\pi}_L(a, b) = c$. Notice that function $\Phi(b)$ is continuously differentiable, strictly decreasing, and that $\Phi(v_s) = v_s$. For all $b \in [v_s, 1]$, also define

$$g(b) = \tilde{\pi}_H(\Phi(b), b) - c.$$

This function is also continuously differentiable with $g(v_s) = 0$. According to the no-resale equilibrium conditions (3), the two equilibrium cutoffs are identified as $b^* \in [v_s, 1]$ and $a^* = \Phi(b^*) \in (0, v_s]$ such that $g(b^*) \leq 0$, with equality if $b^* < 1$. This confirms the existence of the symmetric equilibrium at $a^* = b^* = v_s$. Tan and Yilankaya (2006) show that when F is strictly convex, function $g(\cdot)$ is strictly decreasing around v_s . This implies that the equilibrium conditions are satisfied for at least one pair of asymmetric cutoffs: *Either* there exists a $b^* \in (v_s, 1)$ such that $g(b^*) = 0$ (thus there exists an asymmetric equilibrium with b^* as the high cutoff and $\Phi(b^*)$ as the low cutoff), *or* $g(1) \leq 0$ (thus there exists an asymmetric equilibrium with 1 as the high cutoff and $\Phi(1)$ as the low cutoff).

Now consider the smallest value of b that satisfies these equilibrium conditions: $\bar{b} = \min \{b > v_s : g(b) \leq 0, \text{ with equality if } b < 1\}$. \bar{b} is well-defined since $g(\cdot)$ is continuous and it is *strictly* decreasing for values close enough to v_s . To see that surplus is higher under cutoffs \bar{b} and $\Phi(\bar{b})$, write $\tilde{S}(\Phi(\bar{b}), \bar{b}) - \tilde{S}(v_s, v_s)$ as

$$\begin{aligned} & \int_{v_s}^{\bar{b}} \left(\frac{\partial \tilde{S}(\Phi(b), b)}{\partial b} + \frac{\partial \tilde{S}(\Phi(b), b)}{\partial a} \Phi'(b) \right) db \\ &= \int_{v_s}^{\bar{b}} (-hf(b) \underbrace{[\tilde{\pi}_H(\Phi(b), b) - c]}_{g(b)} - lf(a) \underbrace{[\tilde{\pi}_L(\Phi(b), b) - c]}_0) \Phi'(b) db \\ &= \int_{v_s}^{\bar{b}} -hf(b) g(b) db. \end{aligned}$$

By definition of \bar{b} , the integrand is positive for all $b \in (v_s, \bar{b})$, proving that $S(\Phi(\bar{b}), \bar{b}) > S(v_s, v_s)$. ■

Lemma 1 Consider a low-cutoff bidder with value $w \in [a, b]$ who won the initial auction. Her expected payoff from the resale phase is

$$\beta(w) = b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx. \quad (18)$$

Proof Consider the standard optimal auction problem where the seller's value is $w \in [a, b]$, there are $h \geq 1$ bidders whose valuations are independently distributed on $[0, b]$ according to cdf $\frac{F(\cdot)}{F(b)}$, where $0 \leq a \leq b \leq 1$. Note that (18) is just the expected payoff of the seller, obtained from the standard formulation (see Myerson (1981)) by using a change of variables to incorporate the reserve price. Here we give a heuristic argument as well: The revenue equivalence theorem implies that the continuation payoff for the winner of the initial auction is a continuous function of her valuation and its derivative at each valuation is equal to the probability that the bidder will keep the good at the end of the resale phase. Bidders with $v \geq b$ will not resell the object if they win the initial auction. Accordingly, the continuation payoff of a bidder with valuation b is equal to b . A reseller with value $w \in [a, b]$, who sets her reserve price optimally to $r(w)$, does not sell the object if all h bidders have valuations less than $r(w)$, which happens with probability $\frac{F(r(w))^h}{F(b)^h}$. Accordingly, her continuation payoff is $b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx$. ■

Lemma 2 Consider bidder i who has a higher value than her participation cutoff. Suppose that all bidders except this bidder are following the entry and bidding strategies in (9) and (10). Conditional on participation in the initial auction, it is optimal for bidder i to bid according to (9) and (10).

Proof

- Step 1: Bidding higher than the adjusted values in (9) or (10) is not a profitable deviation.

Suppose that bidder i bids higher than her adjusted value given in (9) or (10). Overbidding will affect this bidder's payoff only in the case that she makes the highest bid and there exists other bidder(s) whose bids are higher than the adjusted value of i . Let j be the highest-value bidder among them and let w be the value of bidder j . Bidder i acquires the good in the initial auction and pays the (adjusted) value of bidder j . Consider the resale stage. Bidder i only knows that she is the winner of the initial auction. In order to

construct an upper bound on the resale payoff of bidder i , suppose that bidder i also learns the identity and the bid (thus, the valuation) of bidder j after the initial auction and is allowed to use this information when designing the resale auction. If $w \geq b$ then bidder i 's optimal resale mechanism is selling the good to bidder j at price w , which brings the same resale revenue to bidder i as she paid in the initial auction. If $w < b$ then both bidder i and bidder j must be low-cutoff bidders. In this case, bidder i 's optimal resale mechanism consists of two steps. First, she will try to resell the good to bidders other than bidder j using an optimal auction with reserve price $r(w)$. If she cannot sell the good in the first step then she can sell it to bidder j at price w . This procedure yields an expected payoff exactly equal to $\beta(w)$, which is the adjusted value of bidder j and the price that bidder i paid in the initial auction. This rules out any positive profit from overbidding.

- Step 2: For a low-cutoff bidder with valuation $v \in [a, b]$, bidding lower than the adjusted value in (9) is not a profitable deviation.

Suppose that bidder i with valuation $v \in [a, b]$ bids lower than her adjusted value $\beta(v) = b - \int_v^b \frac{F(r(x))^h}{F(b)^h} dx$. Underbidding is payoff relevant for this bidder when all high-cutoff bidders have values below b and the highest bidder's bid is lower than the adjusted value of i . Let j be the highest bidder and let w be the value of bidder j . By bidding according to (9), bidder i could have bought the good in the initial auction at price $\beta(w)$ and her expected continuation profit would have been $\beta(v) - \beta(w) = \int_w^v \frac{F(r(x))^h}{F(b)^h} dx > 0$. By underbidding in the initial auction, bidder i can only acquire the good at the resale stage if $v > r(w)$ and all the high-cutoff bidders have values below v . Conditional probability of the latter event is $F(v)^h / F(b)^h$. Using the envelope theorem, the expected resale-stage payoff for the low-cutoff bidder with value $v > r(w)$ will be $\int_{r(w)}^v \frac{F(x)^h}{F(b)^h} dx$. Since $r(x) \geq x$ for $x \in [a, b]$, this is lower than her equilibrium payoff $\beta(v) - \beta(w) = \int_w^v \frac{F(r(x))^h}{F(b)^h} dx$, showing that underbidding is not a profitable deviation for bidder i .

- Step 3: For any bidder with valuation $v > b$, bidding less than v is not a profitable deviation.

Suppose that bidder i with valuation $v > b$ bids lower than v . Underbidding is payoff-relevant for this bidder only if v is the highest realized value and the highest bidder's bid is higher than i 's bid. Let j be the highest bidder and let w be the value of bidder j . If w is higher than b , underbidding is not

a profitable deviation: By bidding less than w , bidder i loses the good and receives zero payoff, whereas by bidding her value she gets the object and obtains the strictly positive payoff $v - w$. If w is lower than b , this implies that the highest bid belongs to a low-cutoff bidder. In this case, underbidding does not effect the probability of receiving the good: if bidder i follows the equilibrium bidding strategy, she receives the good in the initial auction; if she underbids, she receives it in the resale stage. The price to pay in the initial auction is $\beta(w) = b - \int_w^b \frac{F(r(x))^h}{F(b)^h} dx$. The price to pay in the resale depends on the reserve price $r(w)$ and the bids of the other bidders participating in resale. Notice that all these other resale-stage bidders are high-cutoff bidders and they have valuations in $[a, b]$. Depending on whether bidder i is a low or high-cutoff bidder, there may be h or $h - 1$ other potential resale stage bidders. Therefore, the lower bound on the expected resale price is

$$\alpha(b, r(w)) = b - \int_{r(w)}^b \frac{F(x)^{h-1}}{F(b)^{h-1}} dx = b - \int_w^b \frac{F(r(x))^{h-1}}{F(b)^{h-1}} r'(x) dx.$$

It remains to show that this last expression is higher than $\beta(w)$, that is,

$$\int_w^b \frac{F(r(x))^h}{F(b)^h} dx \geq \int_w^b \frac{F(r(x))^{h-1}}{F(b)^{h-1}} r'(x) dx.$$

Multiplying both sides with $F(b)^h$ and rearranging gives

$$\int_w^b F(r(x))^{h-1} [F(r(x)) - F(b) r'(x)] dx \geq 0.$$

We can establish that

$$\begin{aligned} & F(r(x)) - F(b) r'(x) \\ = & [F(r(x)) - F(b)] r'(x) + F(r(x)) [1 - r'(x)] \\ = & \frac{[F(r(x)) - F(b)]}{f(r(x))} f(r(x)) r'(x) + F(r(x)) [1 - r'(x)] \\ = & [x - r(x)] f(r(x)) r'(x) + F(r(x)) [1 - r'(x)] \\ = & \frac{d[x - r(x)] F(r(x))}{dx}, \end{aligned}$$

where the third equality follows from (7). Therefore,

$$\begin{aligned}
& \int_w^b F(r(x))^{h-1} [F(r(x)) - F(b) r'(x)] dx \\
= & \int_w^b F(r(x))^{h-1} d[[x - r(x)] F(r(x))] \\
= & F(r(x))^h [x - r(x)] \Big|_w^b + \int_w^b [r(x) - x] F(r(x)) dF(r(x))^{h-1} \\
= & F(r(w))^h [r(w) - w] + \int_w^b [r(x) - x] F(r(x)) dF(r(x))^{h-1}. \quad (19)
\end{aligned}$$

This last figure is non-negative since $r(x) \geq x$ for all $x \in [a, b]$ and $F(r(x))^{h-1}$ is non-decreasing in x . ■

Lemma 3 *Suppose that all bidders except one are following their equilibrium strategies. For the remaining bidder, the payoff differential between participation in the initial auction and staying out of it is weakly increasing in her valuation.*

Proof It follows from the revenue equivalence theorem that a bidder’s “non-participation” payoff is continuous in her valuation and its derivative is equal to the probability that this bidder will acquire the auctioned object during the resale phase. Her “participation payoff” is continuous in her valuation as well and its derivative is equal to the probability that she receives the good and keeps it after the end of the initial auction and the resale phase. To conclude that the payoff differential is weakly increasing in valuation, it will be sufficient to show that probability of acquiring the good is at least as large for all bidder valuations if the bidder were to participate in the initial auction.

Suppose that the bidder’s valuation is v . If this bidder stays out of the initial auction, she will acquire the object only when the valuations of all the high-cutoff bidders (excluding the bidder in question, in case that she is a high-cutoff bidder) are lower than v and the valuation of the highest low-cutoff bidder (excluding the bidder in question) is between a and $r^{-1}(v)$. Now notice that, when the valuations of the other bidders satisfy this condition, the same bidder would have acquired the object by participating in the initial auction (either by overbidding the highest low-cutoff bidder or at the resale phase) as well.²⁷ This proves that entering in the initial auction does not decrease the probability of acquiring the object for any valuation. ■

²⁷This is analogous to the property (8) in the resale auction discussed in Garratt, Tröger, and Zheng (2009).

Proof of Remark 2

i) $\pi_L(a, b)$ is increasing in a and b for $a > 0$.

Using (9) and (11), we have

$$\pi_L(a, b) = F(a)^{l-1} F(b)^h \beta(a) = F(a)^{l-1} F(b)^h \left[b - \int_a^b \frac{F(r(x))^h}{F(b)^h} dx \right].$$

Let $a > 0$. $\pi_L(a, b)$ is increasing in a , since $\beta(a)$ is increasing in a .

$$\begin{aligned} \frac{\partial \pi_L(a, b)}{\partial b} &= hF(a)^{l-1} \left[F(b)^{h-1} f(b) b - \int_a^b F(r(x))^{h-1} f(r(x)) \frac{\partial r(x)}{\partial b} dx \right] \\ &= hF(a)^{l-1} f(b) \left[F(b)^{h-1} b - \int_a^b F(r(x))^{h-1} r'(x) dx \right] \\ &\geq hF(a)^{l-1} f(b) \left[F(b)^{h-1} b - \int_a^b F(b)^{h-1} dx \right] \\ &= hF(a)^{l-1} f(b) F(b)^{h-1} a > 0, \end{aligned}$$

where the second equality follows from $f(r(x)) \frac{\partial r(x)}{\partial b} = f(b) r'(x)$ (using the implicit function theorem for (7)), and the inequality follows from $F(b) \geq F(r(x))$ and $r'(x) \in (0, 1)$.

ii) $\pi_H(a, b)$ is increasing in a .

From (14),

$$\frac{\partial \pi_H(a, b)}{\partial a} = lF(a)^{l-1} f(a) [F(b)^{h-1} b - \alpha(b, r(a)) + \beta(a)].$$

Using (8), (9), and a change of variables ($x \rightarrow r(x)$),

$$\begin{aligned} \frac{\partial \pi_H(a, b)}{\partial a} &= lF(a)^{l-1} f(a) \left[F(b)^{h-1} b - \frac{1}{F(b)} \int_a^b F(r(x))^{h-1} (F(r(x)) - F(b) r'(x)) dx \right] \\ &\geq lF(a)^{l-1} f(a) \left[F(b)^{h-1} b - \int_a^b F(b)^{h-1} dx \right] \\ &= lF(a)^{l-1} f(a) F(b)^{h-1} a > 0, \end{aligned}$$

where the inequality follows from $F(b) \geq F(r(x))$ and $r'(x) \in (0, 1)$. ■

Proof of Proposition 2 (Existence of Asymmetric Equilibria with Resale)

The proof will use the following lemma.

Lemma 4 *If $0 < a < b$ then $\pi_L(a, b) > \tilde{\pi}_L(a, b)$ and $\pi_H(a, b) < \tilde{\pi}_H(a, b)$.*

Proof Recall that ((1),(2),(11),(14))

$$\begin{aligned}\tilde{\pi}_L(a, b) &= F(a)^{l-1} F(b)^h a \\ \pi_L(a, b) &= F(a)^{l-1} F(b)^h \beta(a) \\ \tilde{\pi}_H(a, b) &= F(b)^{h-1} [F(a)^l b + \int_a^b (b-w) dF(w)^l] \\ \pi_H(a, b) &= F(b)^{h-1} [F(a)^l b + \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l].\end{aligned}$$

Let $0 < a < b$. The first inequality follows from $\beta(a) > a$. The second inequality follows from $b \geq \alpha(b, r(w))$ and $\beta(w) > w$ for $w \in [a, b]$. ■

Proof of Proposition 2. For all $b \in [v_s, 1]$, define

$$\Phi(b) = \begin{cases} a : \pi_L(a, b) = c & \text{if } \pi_L(0, b) < c \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $\Phi(b)$ is continuously differentiable, strictly decreasing whenever it takes positive values, and that $\Phi(v_s) = v_s$. For all $b \in [v_s, 1]$, also define

$$g(b) = \pi_H(\Phi(b), b) - c.$$

This last function is also continuously differentiable with $g(v_s) = 0$. The resale equilibrium conditions are satisfied (with b as the high cutoff and $\Phi(b)$ as the low cutoff) if and only if $g(b) \leq 0$, with equality if $b < 1$.

Recall that $\tilde{\pi}_L(a^*, b^*) = c$. Since function $\pi_L(a, b)$ is increasing in a and is larger than $\tilde{\pi}_L(a, b)$ for $0 < a < b$, it must be that $\Phi(b^*) < a^*$. Consider

$$g(b^*) = \pi_H(\Phi(b^*), b^*) - c < \pi_H(a^*, b^*) - c < \tilde{\pi}_H(a^*, b^*) - c \leq 0.$$

The first inequality follows from monotonicity of π_H in its first argument, and the second one from $\pi_H(a, b) < \tilde{\pi}_H(a, b)$ for $0 < a < b$. Finally, $g(b^*) < 0$ implies the existence of $b^{**} \geq b^*$ (with strict inequality if $b^* < 1$) such that $g(b^{**}) \leq 0$ (with equality if $b^{**} < 1$). Accordingly, there exists an equilibrium

where h bidders use cutoff b^{**} and l bidders use cutoff $a^{**} = \Phi(b^{**})$. To complete the proof, notice that $a^{**} \leq \Phi(b^*) < a^*$. ■

Proof of Proposition 3 (Welfare under Resale)

The proof of the proposition will follow from these lemmas:

Lemma 5 $\frac{\partial S(a,b)}{\partial a} \leq -lf(a) [\pi_L(a,b) - c]$.

Proof

$$\begin{aligned} \frac{\partial S(a,b)}{\partial a} &= -[F(r(a))^h a + \int_{r(a)}^b v dF(v)^h] l F(a)^{l-1} f(a) + lf(a) c \\ &= -lf(a) [F(r(a))^h F(a)^{l-1} a + \int_{r(a)}^b F(a)^{l-1} v dF(v)^h - c]. \end{aligned}$$

Recalling that $\pi_L(a,b) = F(a)^{l-1} F(b)^h \beta(a) = F(a)^{l-1} F(b)^h [b - \int_w^b \frac{F(x)^h}{F(b)^h} dx]$, we need to show

$$F(a)^{l-1} [F(b)^h b - F(r(a))^h a - \int_a^b F(r(x))^h dx] \leq F(a)^{l-1} \int_{r(a)}^b v dF(v)^h, \text{ or}$$

$$F(a)^{l-1} \int_a^b x dF(r(x))^h \leq F(a)^{l-1} \int_a^b r(x) dF(r(x))^h,$$

where the left hand side is obtained by using integration by parts and the right hand side a change of variables. The inequality holds since $x < r(x)$ for all $x < b$. ■

Lemma 6 *If $\frac{vf(v)}{F(v)}$ is weakly increasing in v , then $\frac{\partial S(a,b)}{\partial b} \geq -hf(b) [\pi_H(a,b) - c]$.*

Proof

- **Step 1:** If $\frac{vf(v)}{F(v)}$ is weakly increasing in v , then $F(r(w)) - F(b) r'(w) \geq 0$ for all $w < b$, and the inequality is strict if $w > 0$.

Total differentiation of (7) reveals that

$$r'(w) = \frac{1}{2 + [F(b) - F(r)] \frac{f'(r)}{f(r)^2}},$$

where r equals the optimal reserve price for valuation w . So we need to show

$$F(b) \leq \frac{F(r(w))}{r'(w)} = 2F(r) + [F(b) - F(r)] \frac{f'(r)}{f(r)^2} F(r)$$

$$F(b) - F(r) \leq F(r) + [F(b) - F(r)] \frac{f'(r)}{f(r)^2} F(r),$$

which is identical to

$$\left[1 - \frac{f'(r) F(r)}{f(r)^2}\right] [F(b) - F(r)] \leq F(r)$$

$$\left[\frac{f(r)^2 - f'(r) F(r)}{f(r)}\right] [r - w] \leq F(r).$$

The last line follows from (7). Recall that $r - w \geq 0$. If $f(r)^2 \leq f'(r) F(r)$, then the left hand side is at most zero and the inequality is satisfied. Otherwise, showing

$$\left[\frac{f(r)^2 - f'(r) F(r)}{f(r)}\right] r \leq F(r)$$

is sufficient for the proof. This last inequality is identical to $F(r) f(r) + f'(r) F(r) r - f(r)^2 r \geq 0$. The left hand side is equal to the numerator of the derivative of $\frac{r f(r)}{F(r)}$ with respect to r . The denominator of the derivative is $F(r)^2 > 0$. Therefore if $\frac{v f(v)}{F(v)}$ is weakly increasing in v for all values lower than b , then $F(r(w)) - F(b) r'(w) \geq 0$, concluding the proof of this first step.

- **Step 2:** If $F(r(w)) - F(b) r'(w) \geq 0$ for all $w \in [0, b]$, then $\frac{\partial S(a,b)}{\partial b} \geq -h f(b) [\pi_H(a, b) - c]$.

$$\begin{aligned} \frac{\partial S(a, b)}{\partial b} &= -hbF(b)^{h+l-1} f(b) + hf(b)c \\ &\quad + \int_a^b \left[\begin{array}{c} hbF(b)^{h-1} f(b) + hwF(r(w))^{h-1} f(r(w)) \frac{dr(w)}{db} \\ -hr(w)F(r(w))^{h-1} f(r(w)) \frac{dr(w)}{db} \end{array} \right] dF(w)^l \\ &= -hf(b) [F(b)^{h-1} F(a)^l b - c + \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l]. \end{aligned}$$

where the last equality follows from $f(r(x)) \frac{\partial r(x)}{\partial b} = f(b) r'(x)$ (using the implicit function theorem for (7)).

Recall that

$$\begin{aligned}\pi_H(a, b) &= F(a)^l F(b)^{h-1} b + F(b)^{h-1} \int_a^b [\alpha(b, r(w)) - \beta(w)] dF(w)^l \\ &= F(a)^l F(b)^{h-1} b + \frac{1}{F(b)} \int_a^b \int_w^b F(r(x))^{h-1} [F(r(x)) - F(b) r'(x)] dx dF(w)^l. \quad (20)\end{aligned}$$

We need to show that (20) is larger than or equal to

$$c - \frac{\partial S(a, b)}{\partial b} / hf(b) = F(b)^{h-1} F(a)^l b + \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l.$$

The term $F(b)^{h-1} F(a)^l b$ appears on both sides of this inequality and cancels out. Using equality (19) that we derived for the proof of Lemma 2, the inequality boils down to

$$\begin{aligned}& \frac{1}{F(b)} \int_a^b \left[F(r(w))^h [r(w) - w] + \int_w^b [r(x) - x] F(r(x)) dF(r(x))^{h-1} \right] dF(w)^l \\ & \geq \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l.\end{aligned}$$

Since the value of integral $\int_w^b [r(x) - x] F(r(x)) dF(r(x))^{h-1}$ is non-negative, showing the below inequality would be sufficient for the result:

$$\frac{1}{F(b)} \int_a^b F(r(w))^h (r(w) - w) dF(w)^l \geq \int_a^b (r(w) - w) F(r(w))^{h-1} r'(w) dF(w)^l,$$

which can be rewritten as

$$\int_a^b [F(r(w)) - F(b) r'(w)] F(r(w))^{h-1} (r(w) - w) dF(w)^l \geq 0.$$

This inequality holds since all terms in the integrand are positive for all $w \in (a, b)$ under the hypothesis of the lemma. ■

Proof of Proposition 3. We know from the proof of the Proposition 2 that $g(b^*) < 0$ and there exists b which satisfies the resale equilibrium conditions with $\Phi(b)$. Now consider the smallest such value of b :

$b^{**} = \min \{b \geq b^* : g(b) \leq 0, \text{ with equality if } b < 1\}$. To see that surplus is higher under cutoffs b^{**} and $\Phi(b^{**})$, write $S(\Phi(b^{**}), b^{**}) - S(a^*, b^*)$ as

$$\begin{aligned} & \int_{a^*}^{\Phi(b^*)} \frac{\partial S(a, b^*)}{\partial a} da + \int_{b^*}^{b^{**}} \left(\frac{\partial S(\Phi(b), b)}{\partial b} + \frac{\partial S(\Phi(b), b)}{\partial a} \Phi'(b) \right) db \\ & \geq \int_{\Phi(b^*)}^{a^*} lf(a) [\pi_L(a, b^*) - c] da + \int_{b^*}^{b^{**}} -hf(b) [\pi_H(\Phi(b), b) - c] db \\ & \quad + \int_{b^*}^{b^{**}} -lf(a) [\pi_L(\Phi(b), b) - c] \Phi'(b) db. \end{aligned}$$

The inequality above follows from the previous two lemmas and that $\Phi'(b) \leq 0$. Moreover, $\pi_L(a, b^*) > \pi_L(\Phi(b^*), b^*) = c$ for all $a \in (\Phi(b^*), a^*]$, establishing that the first integral above is strictly positive. The last integral is non-negative since $\pi_L(\Phi(b), b) \geq c$. Finally, the term $[\pi_H(\Phi(b), b) - c]$ in the second integral equals to $g(b)$, which takes non-positive values for $b \in [b^*, b^{**}]$ by definition of b^{**} . ■

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