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Efficient capital allocation in public good networks

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Abstract

A social planner allocates heterogeneous capital, which determines agents' cost of providing local public goods. Given a capital allocation, agents choose equilibrium efforts. Using a first-order approximation, we uncover a tradeoff between allocating productive capital to central and periphery agents, as measured by Bonacich centralities. More productive capital is allocated to central agents if agents' relative risk aversion is below one. Otherwise, it is allocated to the periphery. However, when the planner controls capital allocation and efforts, more productive capital is always allocated to more active agents.

Keywords: Social networks, public goods, Bonacich centrality.

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1 Introduction

An extensive literature on local public goods explores how a network of spillovers shapes equilibrium outcomes (Bloch and Zenginobuz, 2007; Bramoullé and Kranton, 2007; Bramoullé et al., 2014; Elliott and Golub, 2014; Allouch, 2015; Naghizadeh and Liu, 2017). The key insight is that free-riding depends on agents' positions in the network. However, often an important problem is how to incentivize agents to improve efficiency. For example, a social planner controls capital endowments and aims to maximize welfare. Should the central agents who create the highest spillovers be prioritized by the planner?

This paper considers a problem of allocating heterogeneous capital among agents engaged in a game of local public good provision. A capital allocated to an agent determines her cost of providing a public good: more productive capital decreases the agent's marginal cost of effort. Agents benefit from efforts of their neighbors in a social network. Given a capital allocation, agents strategically choose efforts.

Our main result characterizes an efficient capital allocation in terms of the degree of relative risk aversion and Bonacich centralities (Katz, 1953; Bonacich, 1987). Using a first-order approximation, we show that if the risk aversion is below one, then more productive capital is allocated to more central and hence less active agents. Otherwise, more productive capital is allocated to less central, and hence more active agents. This is in contrast to the case when the planner controls a capital allocation and efforts. Then a better capital is always allocated to more active agents.

We contribute to the literature on targeting interventions in networks.[1] The closest papers to ours are Galeotti et al. (2017) and Talamàs and Tamuz (2017). In order to identifying which agents a planner should incentivize the most, Galeotti et al. (2017)

[1]Zenou (2016) provides an excellent literature review on the topic.

propose a new technique based on a singular value decomposition. Talamàs and Tamuz (2017) instead introduce the notion of cycle centrality, and show that the social planner should target the most cycle-central agents. Because both papers use linear-quadratic payoffs, their approaches are not directly applicable to typical models of local public goods. Our analysis relies on a first-order approximation to equilibrium in a network game, the approach pioneered by Acemoglu et al. (2016). Therefore we are forced to assume that differences in capital endowments among agents induce small differences in their marginal costs. However, the advantage of this approach is that it allows us to go beyond linear-quadratic models and accommodate payoff functions used in the literature.

2 Set-up and preliminaries

There are n agents indexed by $N = \{1, \ldots, n\}$. Each agent i chooses effort $e_i \geq 0$, and has constant marginal cost $c_i > 0$. Positive externalities from efforts spread through a network represented by a graph with adjacency matrix $\mathbf{G} \in \mathbb{R}_+^{n \times n}$, where $0 \leq g_{ij} \leq 1$ measures the weight of connection between agents i and j. We assume that benefits flow both ways, letting $g_{ij} = g_{ji}$; by convention $g_{ii} = 0$. Given network $\mathbf{G}$, marginal cost profile $\mathbf{c} = (c_1, \ldots, c_n)$, and effort profile $\mathbf{e} = (e_1, \ldots, e_n)$, payoff of agent i is

$$U_i(\mathbf{c}, \mathbf{e}) = b\left(e_i + \delta \sum_j g_{ij} e_j\right) - c_i e_i.$$

where $\delta \in (0, 1)$ measures the strength of externalities, and b is a continuously differentiable increasing concave function that captures an agent's benefit from public good consumption. Then, for each agent i, the first order condition is

$$b'\left(e_i + \delta \sum_j g_{ij} e_j\right) = c_i. \tag{2.1}$$

For each $x > 0$ define $a(x) = (b')^{-1}(x)$. Then $a(c_i)$ solves (2.1) and is a target local aggregate effort of agent i, whose best reply is $e_i = \max\left\{0, a(c_i) - \delta \sum_j g_{ij} e_j\right\}$. Hence, given $\mathbf{c}$, an interior Nash equilibrium satisfies

$$\mathbf{e}(\mathbf{c}) = (\mathbf{I} + \delta \mathbf{G})^{-1} \mathbf{a}(\mathbf{c}), \tag{2.2}$$

where $\mathbf{a}(\mathbf{c}) = (a(c_1), \ldots, a(c_n))$. Ballester et al. (2006) show that the inverse exists and there is a unique equilibrium if $\delta < 1/\lambda_{max}$, where λ_{max} is the largest eigenvalue of $\mathbf{G}$. Throughout the paper we assume the existence of a unique interior equilibrium. For each pair i and j, we let ϕ_{ij} denote the corresponding entry of matrix $(\mathbf{I} + \delta \mathbf{G})^{-1}$.

For network $\mathbf{G}$ and scalar α, the *Bonacich centrality* (Bonacich, 1987) of agent i is defined as $v_i(\mathbf{G}, \alpha) = \left[(\mathbf{I} - \alpha \mathbf{G})^{-1} \mathbf{G} \mathbf{1}\right]_i$, where $\mathbf{1}$ is a vector of ones, and $[\mathbf{x}]_i$ is the i's coordinate of vector $\mathbf{x}$. Ballester et al. (2006) show that equilibrium under symmetric cost profile $\mathbf{c}^0 = (c, \ldots, c)$, can be expressed in terms of Bonacich centralities as $e_i(\mathbf{c}^0) = a(c)(1 - \delta v_i(\mathbf{G}, -\delta))$ for each agent i. Hence, the most central agents are the least active, and the least central are the most active.

The Arrow-Pratt measure of relative risk aversion of benefit function b at x is given by

$$\eta(x) = -\frac{x b''(x)}{b'(x)}. \tag{2.3}$$

The measure is related to the marginal cost elasticity of target local aggregate effort $a(c_i)$. Indeed, the absolute value of elasticity of a target local aggregate effort of agent i

is given by $-a'(c_i)c/a(c_i)$. Using (2.1) we can rewrite it as $1/\eta(a(c_i))$. Hence, the more risk averse an agent is, the less her target local aggregate effort responds to changes in her marginal cost.

3 Efficient capital allocation under equilibrium efforts

We study efficient capital allocation when agents choose efforts strategically. A social planner allocates heterogeneous capital indexed by $M = \{1, \ldots, n\}$ among the agents.[2] A capital allocated to an agent determines the agents marginal cost. If agent i is allocated capital k, then she has marginal cost $c_i = c^k$.[3] We let $c^1 < c^2 < \cdots < c^n$, that is, capital 1 is the most productive, and capital n is the least productive. A capital allocation is a bijection $\sigma : N \to M$, where $\sigma(i)$ denotes a capital allocated to agent i. For allocation σ, we denote the corresponding profile of marginal costs by $\mathbf{c}^\sigma = (c^{\sigma(1)}, \ldots, c^{\sigma(n)})$. Given marginal cost profile $\mathbf{c}$, let $\mathbf{e}(\mathbf{c})$=$(e_1(\mathbf{c}), \ldots, e_n(\mathbf{c}))$ be the unique interior equilibrium of the n-player strategic game. The planner chooses capital allocation σ to maximize equilibrium social welfare $W(\mathbf{c}^\sigma, \mathbf{e}(\mathbf{c}^\sigma)) = \sum_i U_i(\mathbf{c}^\sigma, \mathbf{e}(\mathbf{c}^\sigma))$.

To provide a sharp characterization of the efficient allocation we follow the approach of Acemoglu et al. (2016) and use a first-order approximation to equilibrium welfare. In order to guarantee that the approximation captures a dominant effect of changes in capital endowments on welfare, we assume that the differences in agents' marginal costs are small. We evaluate a linear approximation to equilibrium welfare at a symmetric marginal cost profile.

[2]Alternatively, we could consider a continuous model where a social planner chooses a division of a small additional unit of capital among agents. However, given the approach in this paper, continuous formulation produces qualitatively the same results.

[3]Assuming that the number of agents is the same as the number of units of capital is without loss of generality. In what follows, we use subscripts for agents, and superscripts for capital.

Proposition 1. *Suppose σ^* maximizes the first-order approximation to equilibrium welfare around $\mathbf{c}^0 = (c, \ldots, c)$, where $c = c^1$. We have two cases:*

(i) If $\eta(a(c)) < 1$, then agents with higher Bonacich centralities are allocated more productive capital: $v_i(\mathbf{G}, -\delta) > v_j(\mathbf{G}, -\delta)$ implies $\sigma^*(i) < \sigma^*(j)$.

(ii) If $\eta(a(c)) > 1$, then agents with higher Bonacich centralities are allocated less productive capital: $v_i(\mathbf{G}, -\delta) > v_j(\mathbf{G}, -\delta)$ implies $\sigma^*(i) > \sigma^*(j)$.

Proof. First, notice that $\sum_i b(a(c^{\sigma(i)}))$ is invariant under σ. Then, choosing σ to maximize equilibrium welfare is equivalent to choosing σ to minimize the equilibrium total cost, $\sum_i c^{\sigma(i)} e_i(\mathbf{c}^\sigma)$. Consider the first-order approximation to equilibrium total cost around $\mathbf{c}^0$. We have:

$$\sum_i c_i e_i(\mathbf{c}^\sigma) \approx \sum_i c_i e_i(\mathbf{c}^0) + \nabla \left[\sum_i c_i e_i(\mathbf{c}) \right] \Bigg|_{\mathbf{c}=\mathbf{c}^0} \cdot (\mathbf{c}^\sigma - \mathbf{c}^0),$$

where $\nabla \left[\sum_i c_i e_i(\mathbf{c})\right]\big|_{\mathbf{c}=\mathbf{c}^0}$ is the gradient of the total cost evaluated at $\mathbf{c}^0$. For agent i we have:

$$\begin{aligned}
\left.\frac{\partial \sum_j c_j e_j(\mathbf{c})}{\partial c_i}\right|_{\mathbf{c}=\mathbf{c}_0} &= e_i(\mathbf{c}^0) + \sum_j c_j \left.\frac{\partial e_j(\mathbf{c})}{\partial c_i}\right|_{\mathbf{c}=\mathbf{c}^0}, \\
&= a(c) \sum_j \phi_{ij} + \left.\sum_j c_j \phi_{ji} a'(c_i)\right|_{\mathbf{c}=\mathbf{c}^0}, \\
&= \Big(a(c) + c a'(c)\Big) \sum_j \phi_{ji}, \\
&= a(c) \left(1 - \frac{1}{\eta(a(c))}\right) \left(1 - \delta v_i(\mathbf{G}, -\delta)\right),
\end{aligned}$$

where the second line follows from (2.2) and ϕ_{ij} being the corresponding entry of matrix $(\mathbf{I}+\delta\mathbf{G})^{-1}$, the third line from $\phi_{ij} = \phi_{ji}$ for each pair i and j by symmetry of $(\mathbf{I}+\delta\mathbf{G})^{-1}$, and the last line from (2.3) and rewriting the Bonacich centrality definition. The result follows from evaluating the sign of the derivative. □

The intuition is as follows. Allocating more productive capital to an agent has two effects on social welfare. The first one is direct: it reduces the agent's cost of effort. The direct effect is the highest for the least central agents, who provide the highest effort. The second is an indirect network effect: the reduction in the marginal cost induces the agent to require more effort around her, resulting in some agents decreasing, and some increasing their efforts. The network effect is the highest for the most central agents, who also free-ride the most, exerting the lowest effort. If the relative risk aversion is below one, the direct effect dominates, and more productive capital is allocated to less central agents. Otherwise, the network effect dominates, and more productive capital is allocated to more central agents. Hence, the social planner faces a tradeoff between reducing the costs of the most active agents and inducing the highest spillovers by incentivizing the central, but the least active agents.

4 Efficient capital allocation under the first best efforts

The above analysis assumes that, given the capital endowments, agents choose an equilibrium effort profile. Here we provide a useful benchmark where the social planner controls both, a capital allocation and agents' efforts.

Proposition 2. *(Naghizadeh and Liu, 2017) An interior welfare maximizing effort profile is given by*

$$\mathbf{e}^*(\mathbf{c}) = (\mathbf{I} + \delta\mathbf{G})^{-1}\,\bar{\mathbf{a}}(\mathbf{c}), \tag{4.1}$$

where $\bar{\mathbf{a}} = (\bar{a}_1(\mathbf{c}), \ldots, \bar{a}_n(\mathbf{c}))$ *is such that* $b'(\bar{a}_i(\mathbf{c})) = [(\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{c}]_i$ *for each agent* i.

Note that the expression for the efficient profile differs from (2.2) only by vector $\bar{\mathbf{a}}(\mathbf{c})$, which represents the adjusted target local aggregate efforts of agents. Whereas $\mathbf{a}(\mathbf{c})$ reflects only individual incentives, $\bar{\mathbf{a}}(\mathbf{c})$ accounts for the existence of positive spillovers among agents. In particular, agents with higher Bonacich centralities are assigned higher values of $\bar{a}_i(\mathbf{c})$.

Proposition 3. *Suppose that, given a capital allocation, agents choose the efficient effort profile. Then allocation σ^* maximizes the first-order approximation to equilibrium welfare around $\mathbf{c}_0$ if $e_i^*(\mathbf{c}^0) > e_j^*(\mathbf{c}^0)$ implies $\sigma^*(i) < \sigma^*(j)$.*

Proof. Consider the first-order approximation to equilibrium welfare around $\mathbf{c}_0$,

$$W(\mathbf{c}^\sigma, \mathbf{e}^*(\mathbf{c}^\sigma)) \approx W(\mathbf{c}^0, \mathbf{e}^*(\mathbf{c}^0)) + \nabla \left[\sum_i b\left(\bar{a}_i(\mathbf{c})\right) - c_i e_i^*(\mathbf{c})\right]\Bigg|_{\mathbf{c}=\mathbf{c}^0} (\mathbf{c}^\sigma - \mathbf{c}^0).$$

For each agent i we have

$$\begin{aligned}
\frac{\partial \left[\sum_j b\left(\bar{a}_j(\mathbf{c})\right) - c_j e_j^*(\mathbf{c})\right]}{\partial c_i}\Bigg|_{\mathbf{c}=\mathbf{c}^0} &= \left[\sum_j b'(\bar{a}_j(\mathbf{c}))\frac{d\bar{a}_j'(\mathbf{c})}{dc_i} - c_j\frac{de_j^*(\mathbf{c})}{dc_i}\right]\Bigg|_{\mathbf{c}=\mathbf{c}^0} - e_i^*(\mathbf{c}^0), \\
&= \left[\sum_j\sum_k \left(\phi_{jk}c_k\frac{\phi_{ji}}{b''(\bar{a}_j(\mathbf{c}^0))} - \phi_{jk}c_j\frac{\phi_{ki}}{b''(\bar{a}_k(\mathbf{c}^0))}\right)\right] - e_i^*(\mathbf{c}^0), \\
&= -e_i^*(\mathbf{c}^0),
\end{aligned}$$

where the second line follows from (4.1), and the third line from cancelling out the terms in the double sum. □

The result is intuitive: when an effort profile is controlled by the social planner, only the direct cost reduction matters. Therefore, in contrast to Proposition 2, more productive capital is always allocated to more active agents.

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