NAZARBAYEV UNIVERSITY

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Labyrinth Chaos: Non-Hamiltonian, Conservative, Elegant, without any attractors!

Presented by Vasileios Basios

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- https://arxiv.org/abs/2004.14336

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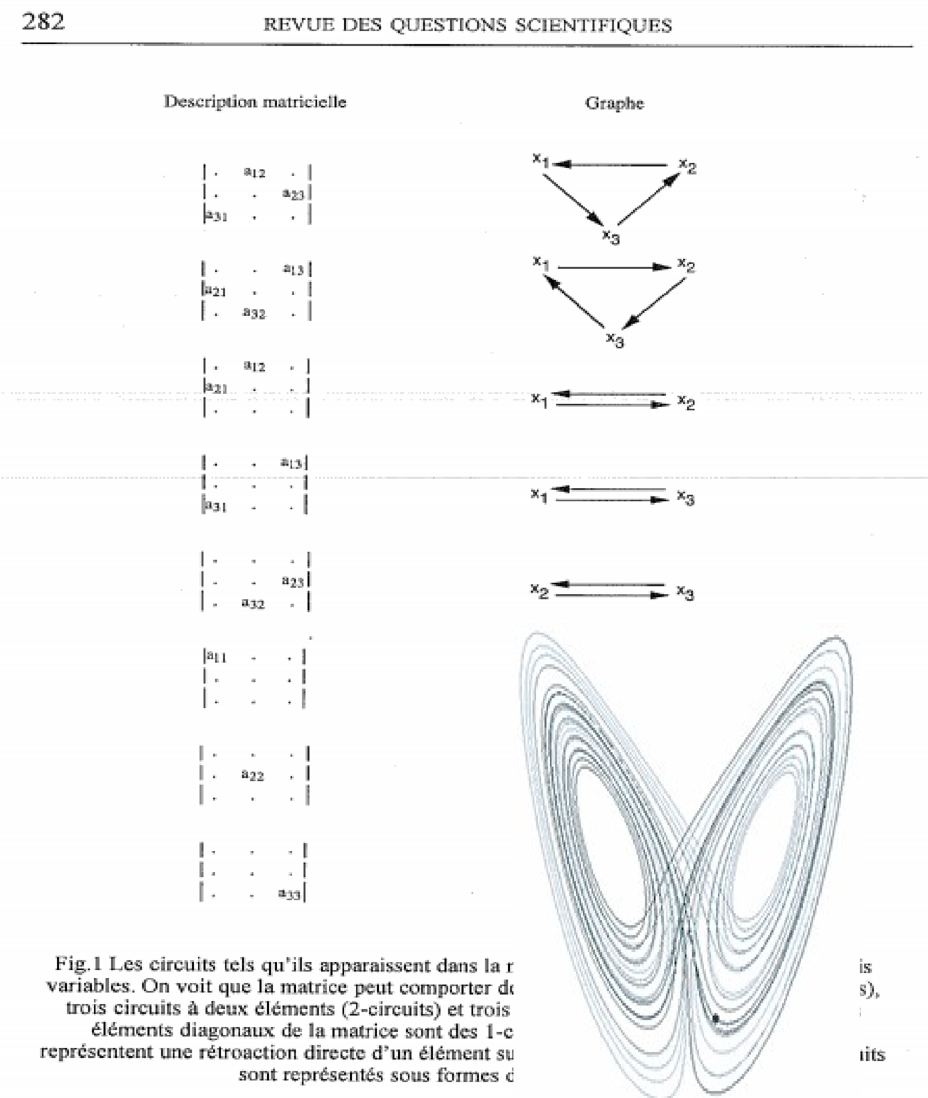
Feedback Circuits, Cycles, ogic Topology



Description matricielle

Graphe

Fig.1 Les circuits tels qu'ils apparaissent dans la r
variables. On voit que la matrice peut comporter de
trois circuits à deux éléments (2-circuits) et trois
éléments diagonaux de la matrice sont des 1-c
représentent une rétroaction directe d'un élément su
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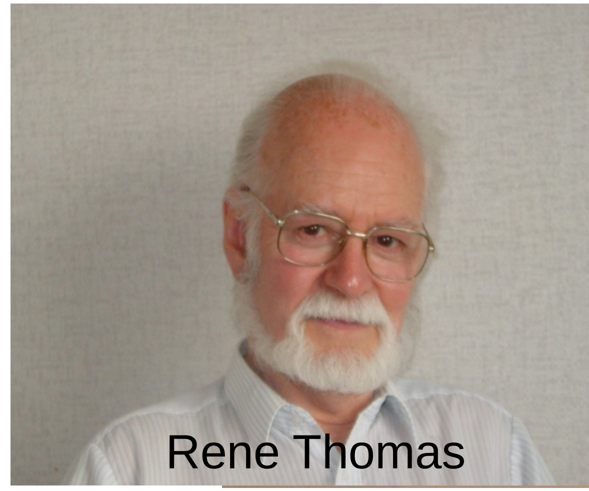
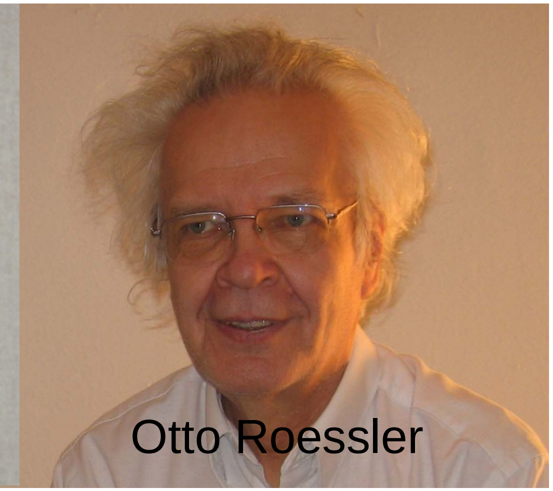
Int. J. of Bifurcation and Chaos **23**:09.(2013)



Rene Thomas



Otto Roessler

Gregoire Nicolis

Leonid Shilnikov

Journal of Theoretical Biology 474 (2019) 42–47

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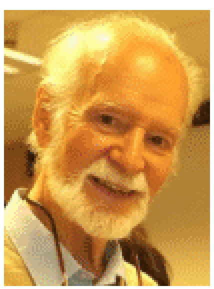
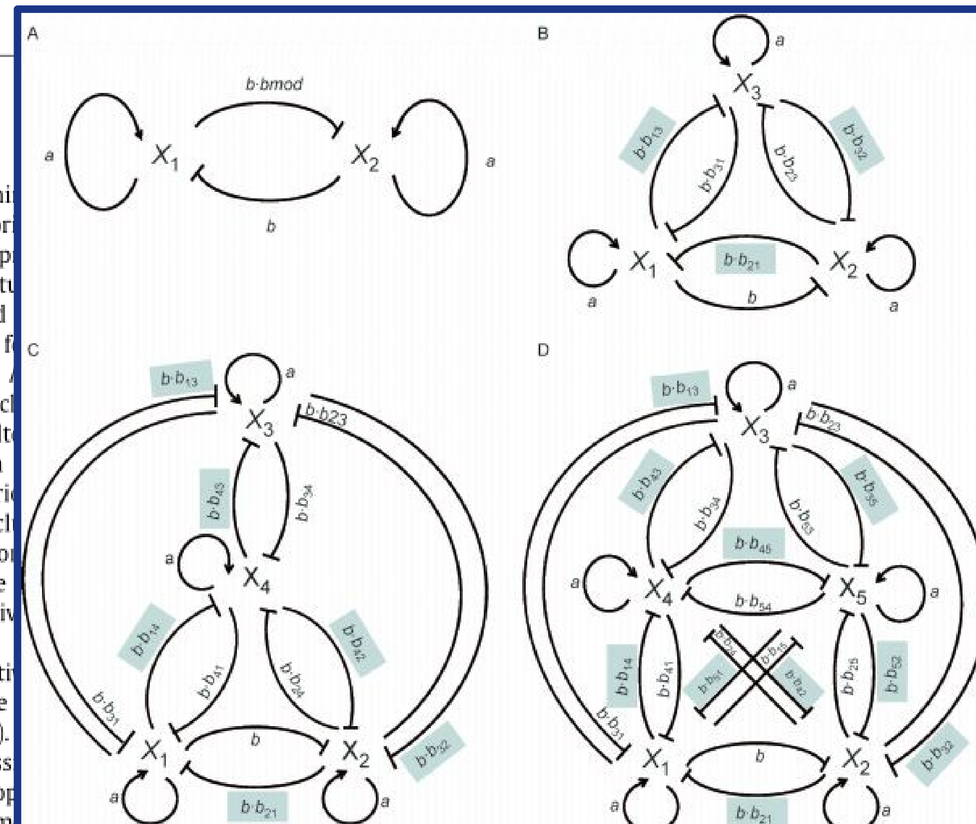
journal homepage: www.elsevier.com/locate/jtb

Introduction

Prologue to the special issue of JTB dedicated to the memory of René Thomas (1928–2017)☆

A journey through biological circuits, logical puzzles and complex dynamics





Born in 1928 in Brussels, Belgium, René Thomas studied Biochemistry and Zoology at the Free University of Brussels (ULB), which remained his academic home throughout most of his amazingly diversified scientific career. The originality and quality of Thomas research led him to various prestigious awards and honours, including his election at the Royal Academy of Science of Belgium (1975), the Francqui Prize (1975), the FNRS Quinquennal Prize (1981-85), and the Golden Medal of the French Academy of Sciences (1999).

Thomas' PhD work dealt with the biophysical and biochemical study of nucleic acids, under the supervision of the Belgian embryologist Jean Brachet. He discovered that the UV absorption of native DNA is much lower than expected from the theoretical spectrum computed from the extinction coefficients of its component nucleotides. Furthermore, he showed that mild treatments, such as lower or higher pH, higher temperature, or lower ionic strength, lead to UV absorption spectra matching theoretical prediction. As these treatments preserve the covalent inter-nucleotide

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Thomas researc
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discoveries, incl
repressor ("Tho
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rence of positiv
Dambly et al.,
solely on negati
products of the
bacteria *E. coli*).

The progress
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the behaviour of such networks. Hence, he looked for means to formalise such regulatory networks and rigorously analyse their dynamical properties. This led him to consider Boolean algebra, which he initially learned by attending classes by Jean Florine and interacting with Philippe Van Ham (see his testimony below) at

- By the term **circuits** we refer to those sets of terms of the **Jacobian matrix** of the dynamical system whose row and column indices are in circular permutation

- Circuits are positive or negative according to the sign of the product of their terms.

Non-linear Arabesques
A1n3: (chaos with just one cubic nonlinearity)

$$\frac{dx}{dt} = y^3 - z$$
$$\frac{dy}{dt} = z - x,$$
$$\frac{dz}{dt} = x - y$$

$$J = \begin{pmatrix} 0 & 3y^2 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Steady state	Eigenvalues of steady state
$(-1,-1,-1)$	$-0.3882\ldots,\ 0.1941\ldots \pm 2.2612\ldots\imath$
$(0,0,0)$	$0.4533\ldots,\ -0.2266\ldots \pm 1.4677\ldots\imath$
$(1,1,1)$	$-0.3882\ldots,\ 0.1941\ldots \pm 2.2612\ldots\imath$

Frontiers:

F1: sign of the product of the (real) eigenvalues of J changes: sign(P)= sign($(-1)^n$ det[J])

F2: sign of the real / imaginary part changes

F4: (boundary) eigenvalues change from real to complex

$$F_1 : \quad -1 + 27x^2y^2z^2 = 0$$
$$F_2 : \quad -1 + 27x^2y^2z^2 = 0$$
$$F_4 : 4(3x^2 + 3y^2 + 3z^2)^3 + 27(1 - 27x^2y^2z^2)^2 = 0$$

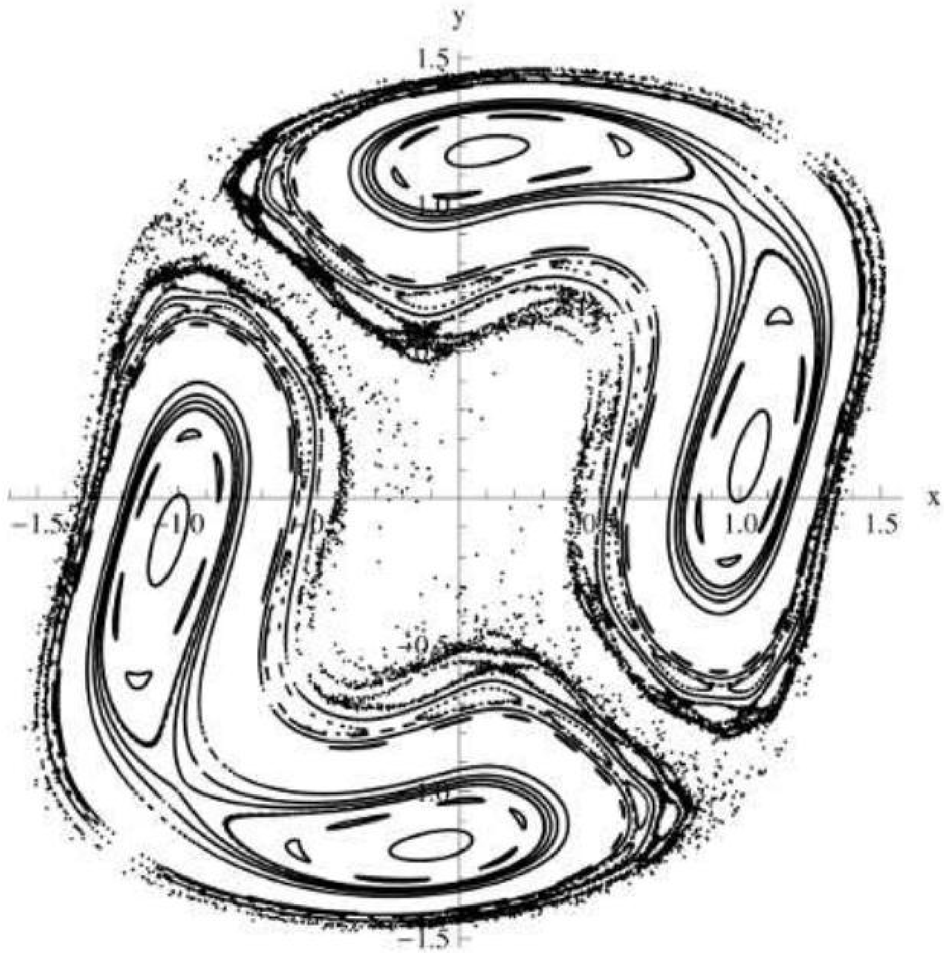
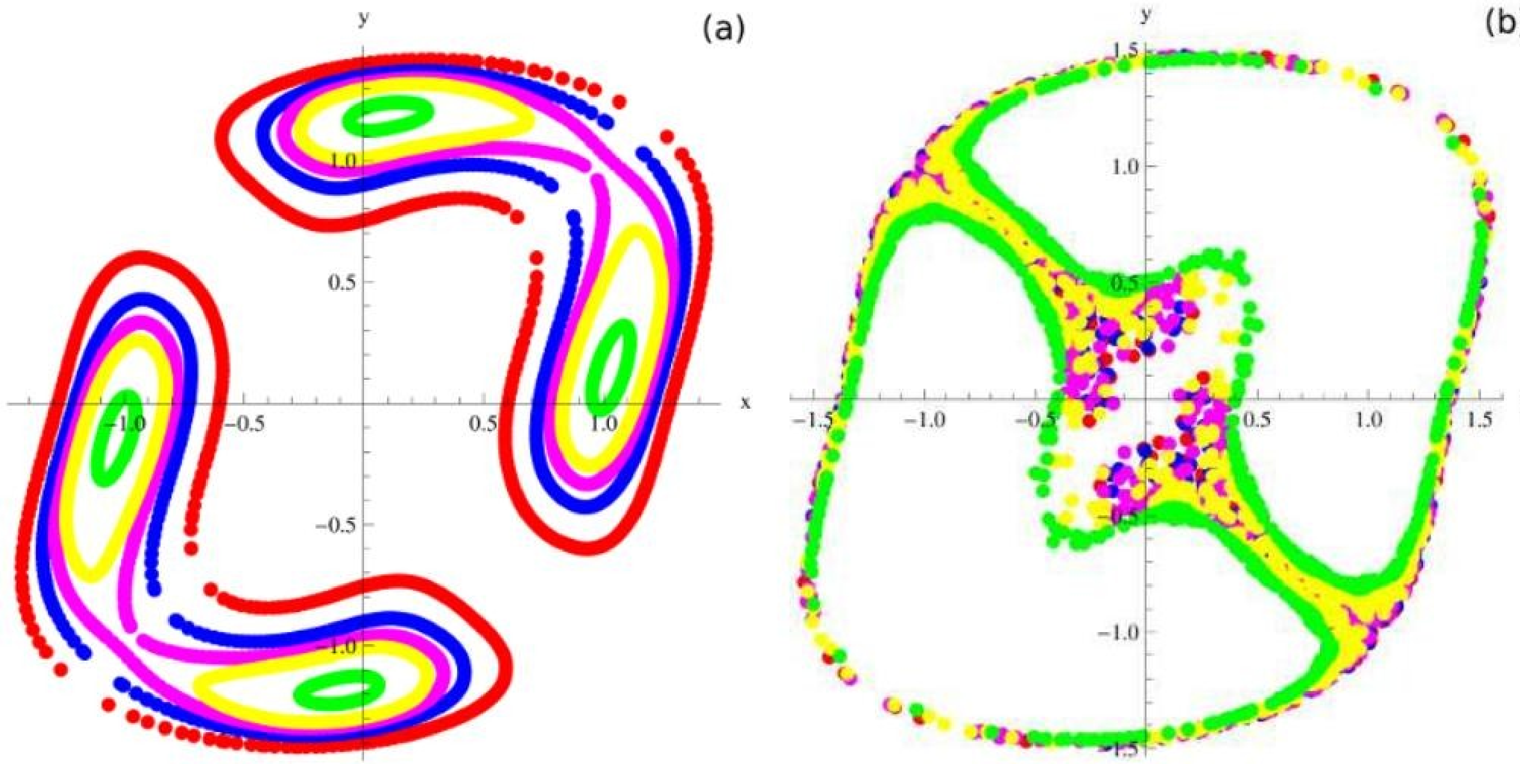


Fig. 7. PSS of points (x, y) satisfying $z = 0$ of the state-space of system (21) as a juxtaposition for $z \geq 0$ and $z \leq 0$ due to its point-symmetric property.



(a)
y
x
1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0
(b)
y
x
1.5
1.0
0.5
−1.5
−1.0
−0.5
0.5
1.0
1.5
−1.0
-1.5

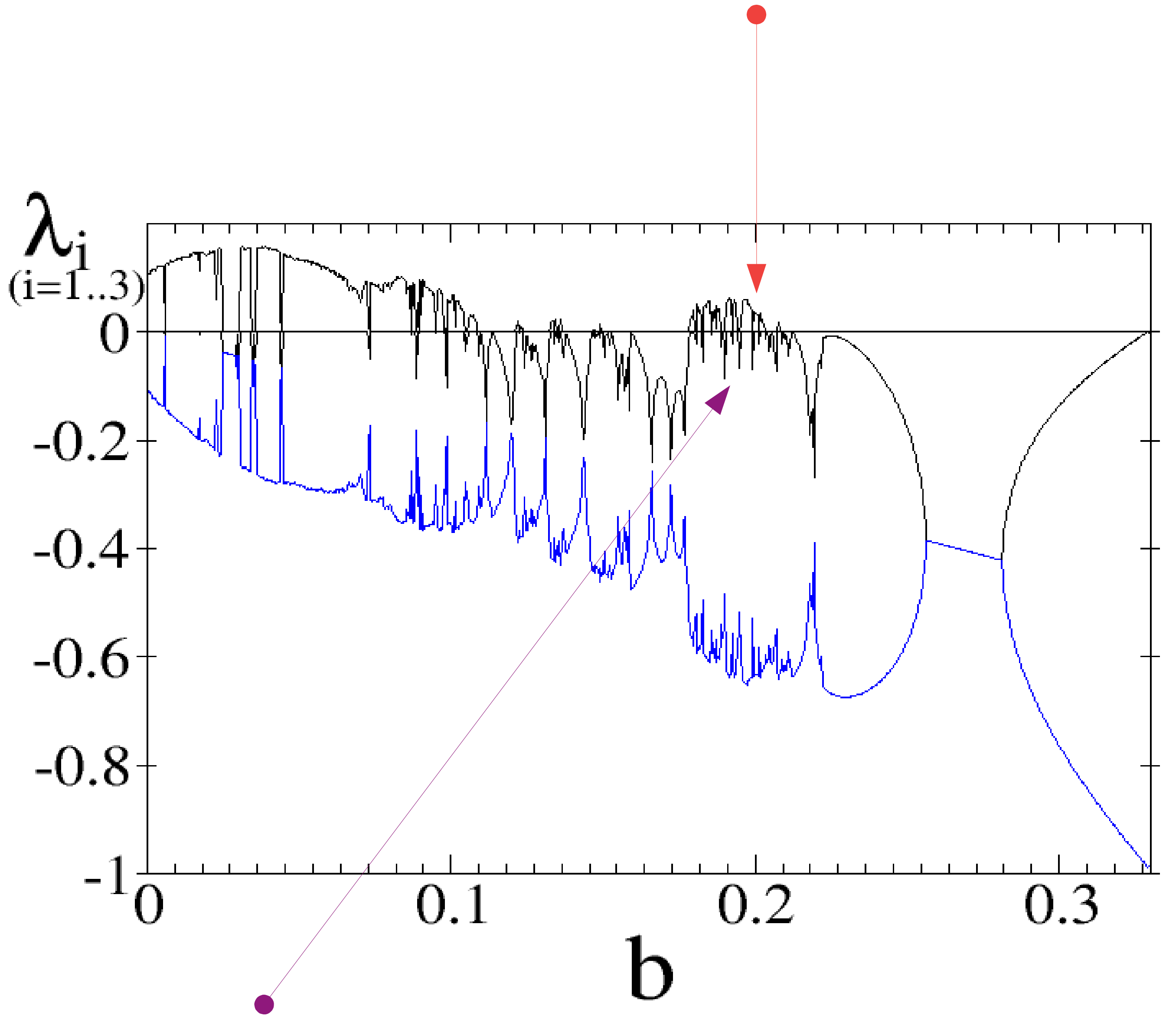
3D- Labyrinth Chaos Thomas-Rőssler Systems

$$\begin{aligned} \frac{dx}{dt} &= -bx + \sin(y) \\ \frac{dy}{dt} &= -by + \sin(z) \\ \frac{dz}{dt} &= -bz + \sin(x) \end{aligned}$$

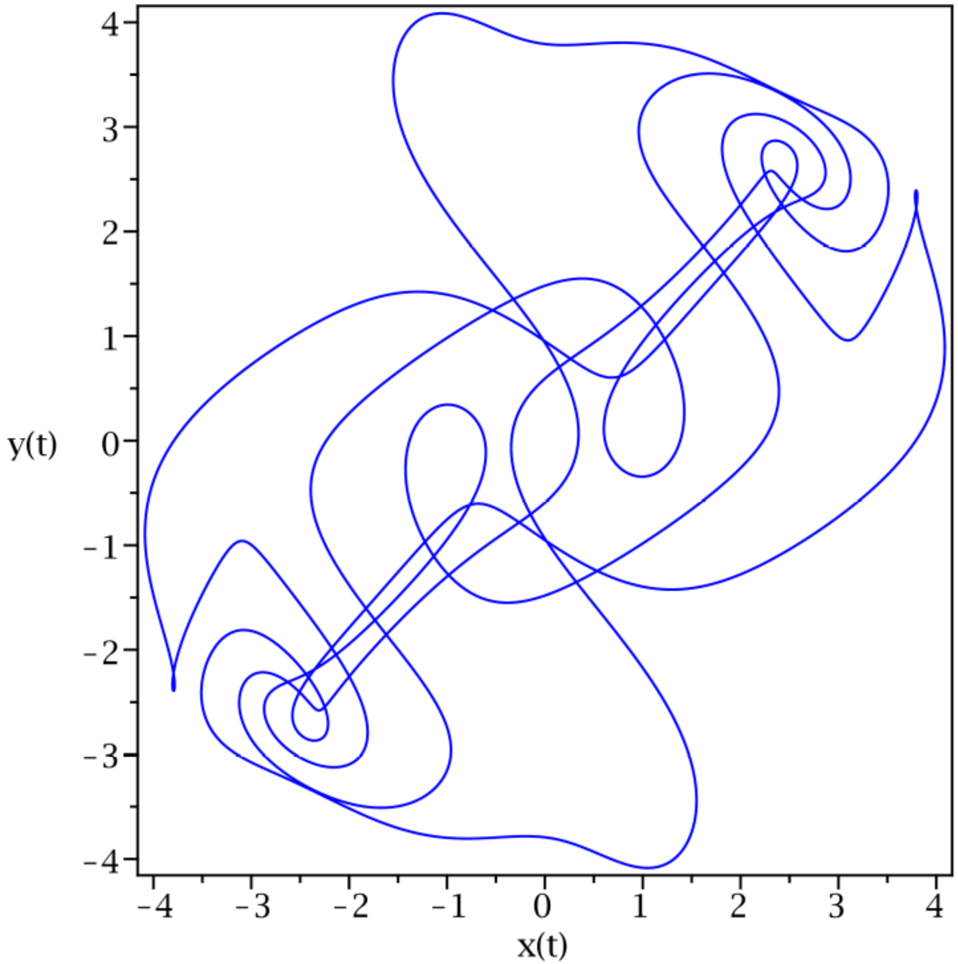
$$\mathbf{J} = \begin{bmatrix} -b & \cos(y_1) & 0 \\ 0 & -b & \cos(z_1) \\ \cos(x_1) & 0 & -b \end{bmatrix}$$

Lyapunov exponents for the 3D Labyrinth Chaos

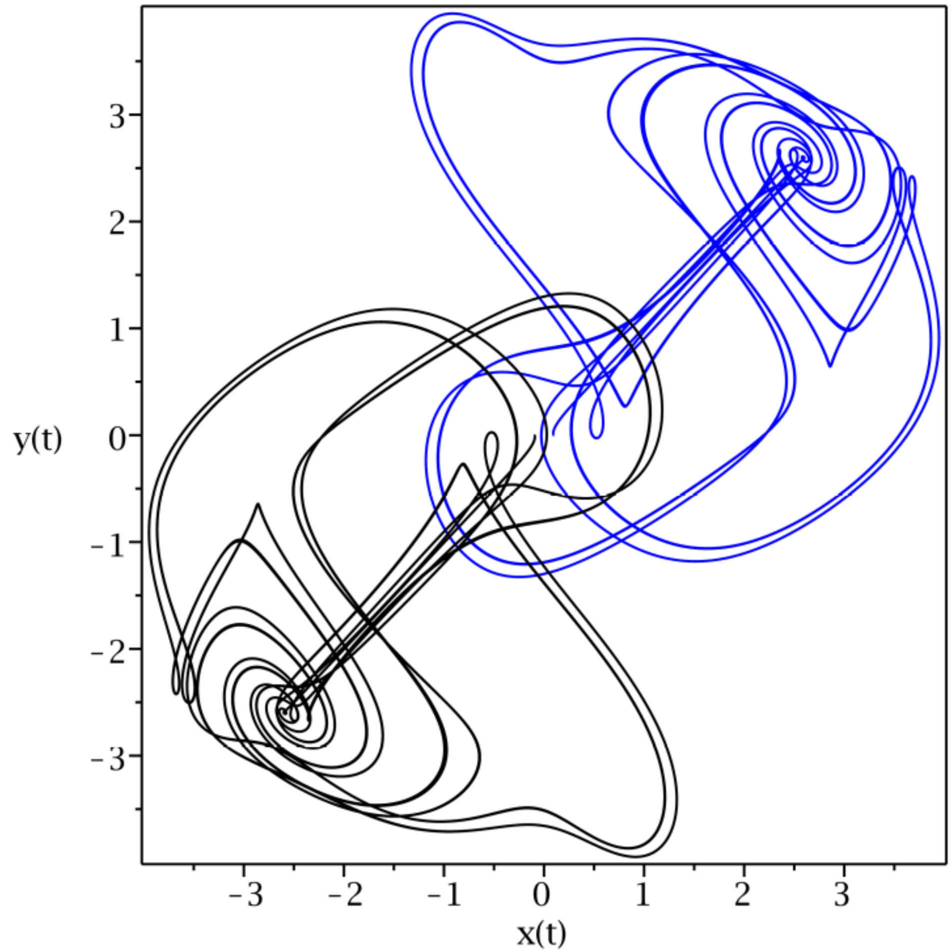
$$\frac{dx}{dt} = -bx + \sin(y)$$
$$\frac{dy}{dt} = -by + \sin(z)$$
$$\frac{dz}{dt} = -bz + \sin(x)$$



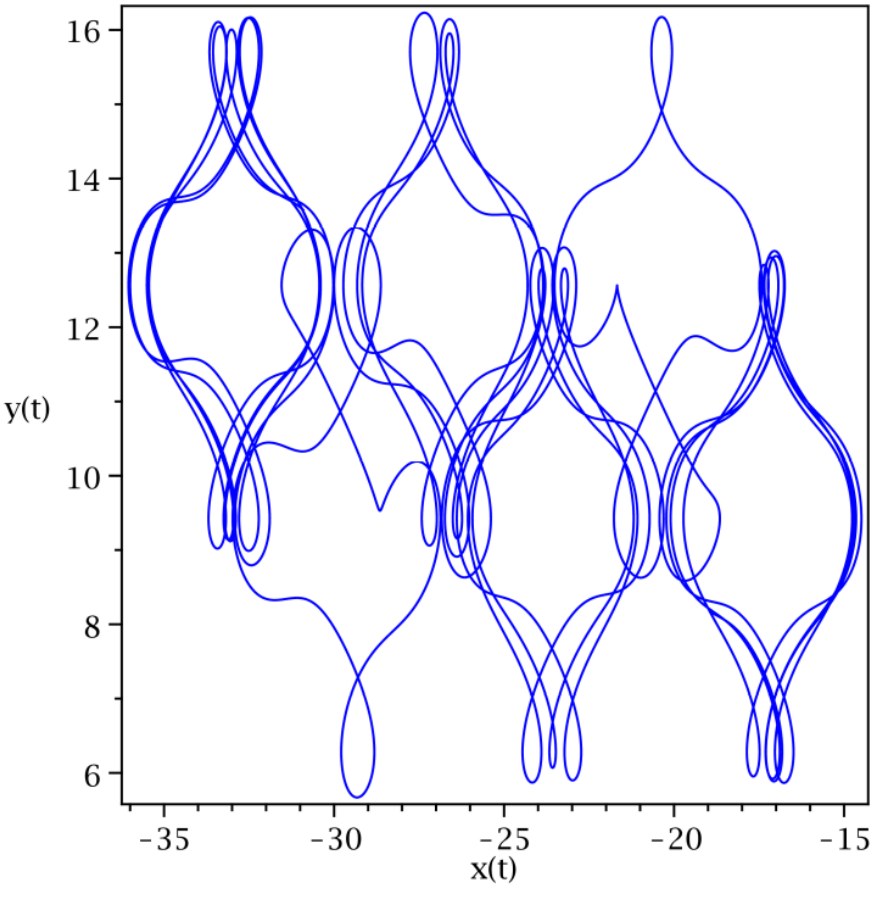
b=0.19
periodic orbit

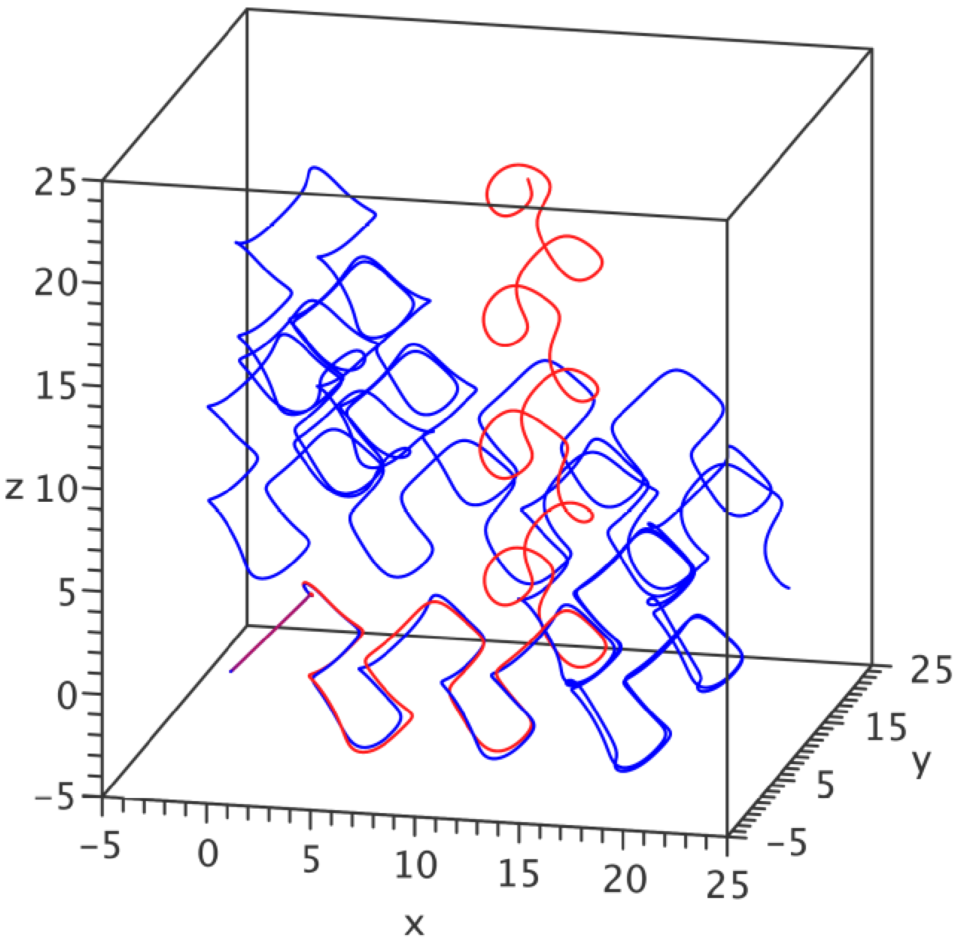


b=0.2
coexisting attractors



b = 0 : Labyrinth Chaos





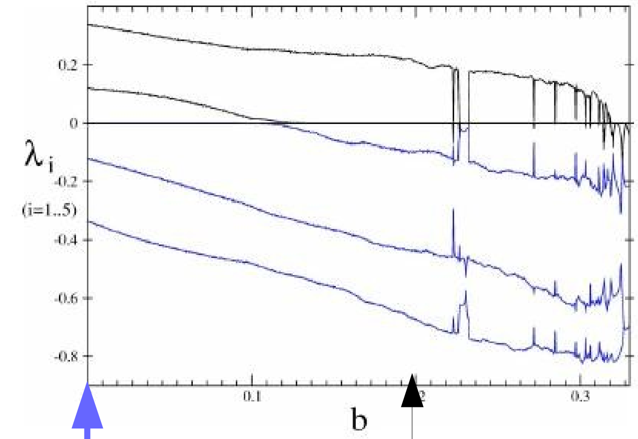
Labyrinth Chaos (N=5)

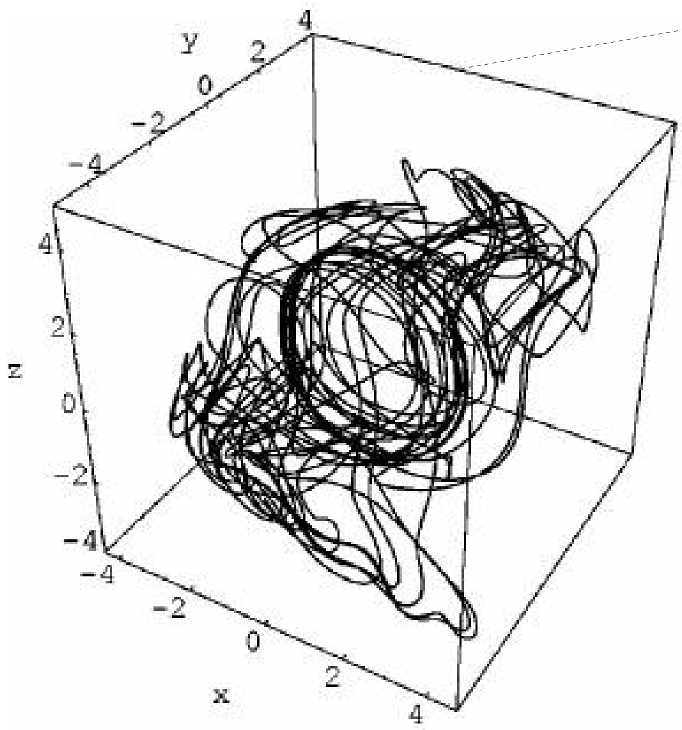
$$\dot{x}_i = \sin(x_\phi) - b x_i ,$$

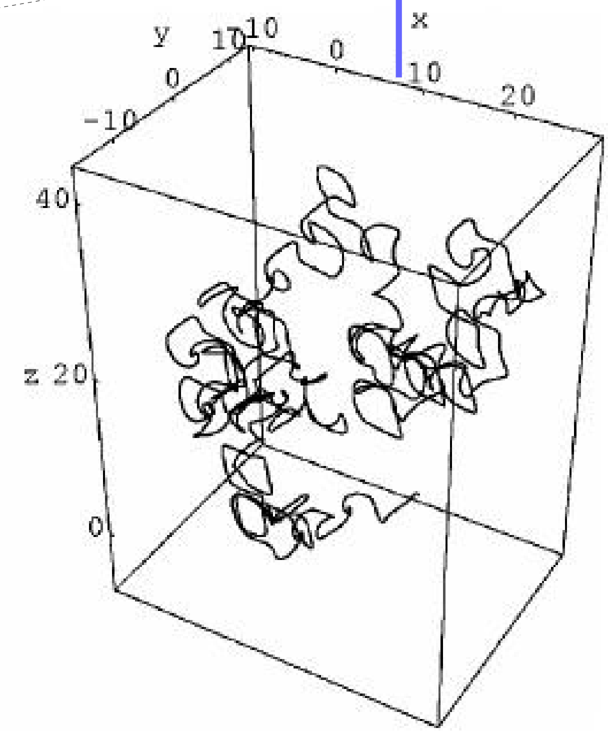
$$\phi = (i \mod n) + 1$$

With n = 1, … N = 2m+1

Is there a hidden "Hamiltonian"-like structure?
Implicitly assumed yes … but







Int. J. of Bif. and Chaos, 17, 6 (2007), pp 2097–2108
"LABYRINTH CHAOS", J. C. Sprott, K. E. Chlouverakis

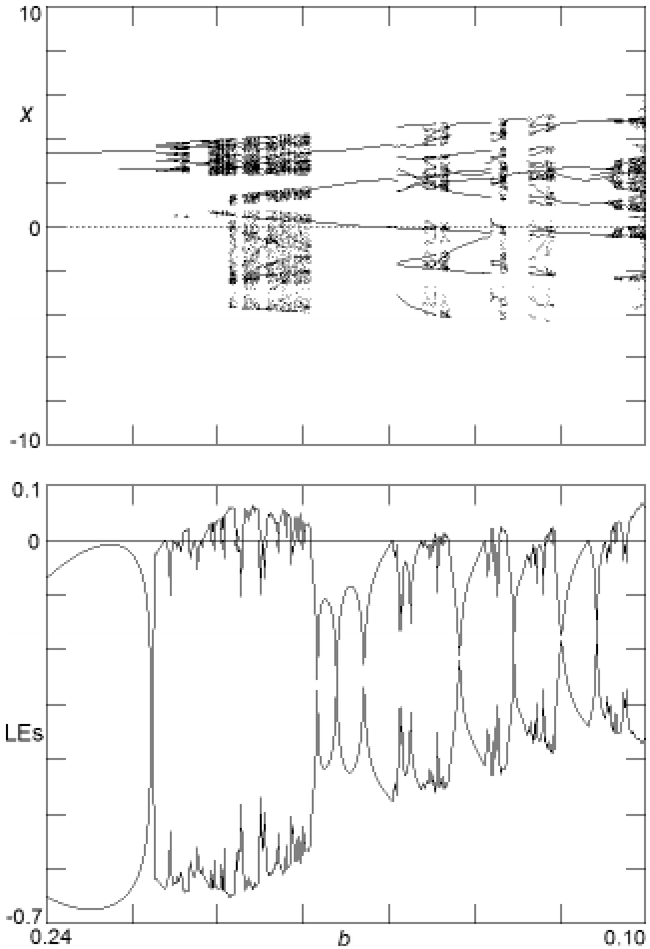


Fig. 2. Bifurcation diagram (local maximum of x) and Lyapunov exponents versus b showing the route to chaos in greater detail.

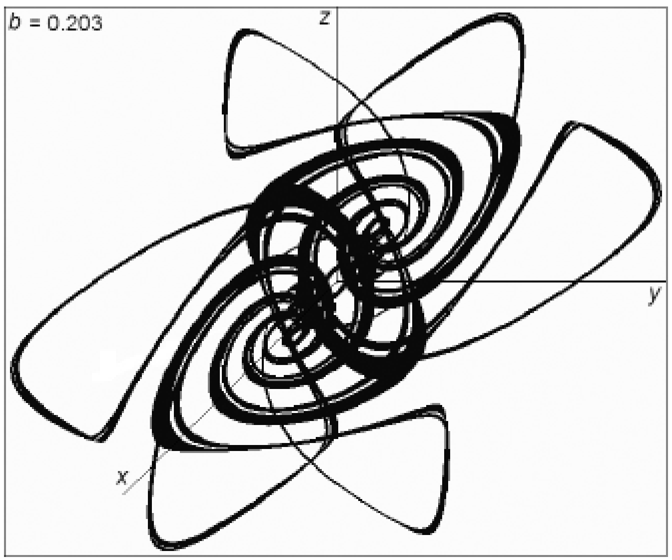


Fig. 8. Six coexisting strange attractors at $b = 0.203$.

SIX coexisting attractors!
b=0.203
N=3

Int. J. of Bif. and Chaos, 17, 6 (2007), pp 2097–2108, *“LABYRINTH CHAOS”*, J. C. Sprott, K.E. Chlouverakis

Invariant Density:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x}\sin y + \frac{\partial f}{\partial y}\sin z + \frac{\partial f}{\partial z}\sin x = 0$$

whose solution is $f(x, y, z) = \text{constant}$. Eq

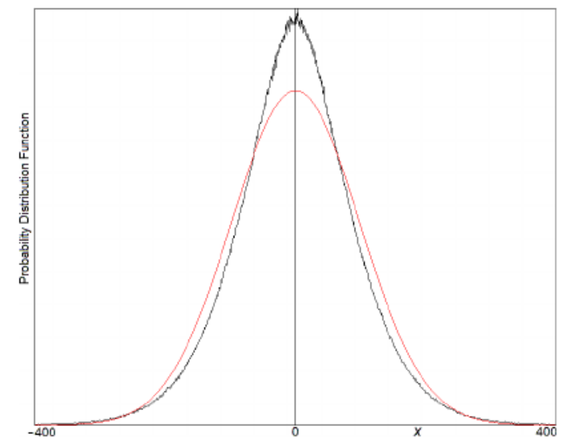


Fig. 13. Probability distribution function of x for 5×10^6 initial conditions near the origin after a time of 4×10^3. The red curve is a Gaussian distribution with the same standard deviation and area.

Fractional Brownian Motion
Hurst Exponent ~ 0.61 > 1/2

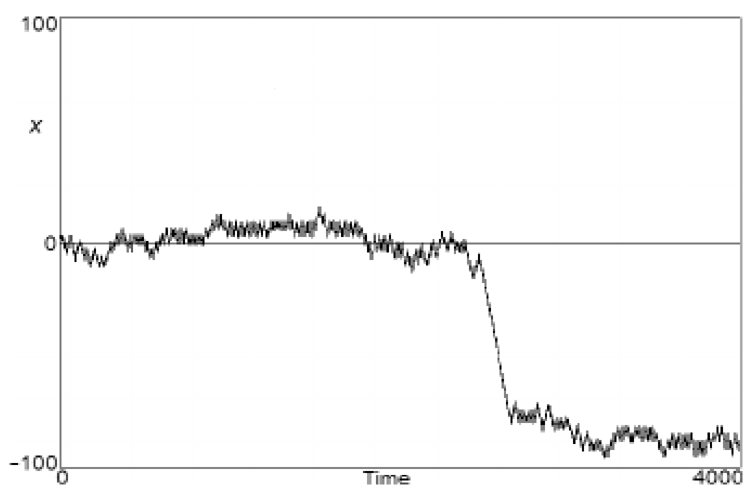


Fig. 14. Projection of the trajectory onto the x-axis showing an example of intermittency where the trajectory approaches the quasiperiodic region with initial conditions (0.05, 0.09, 0.05).

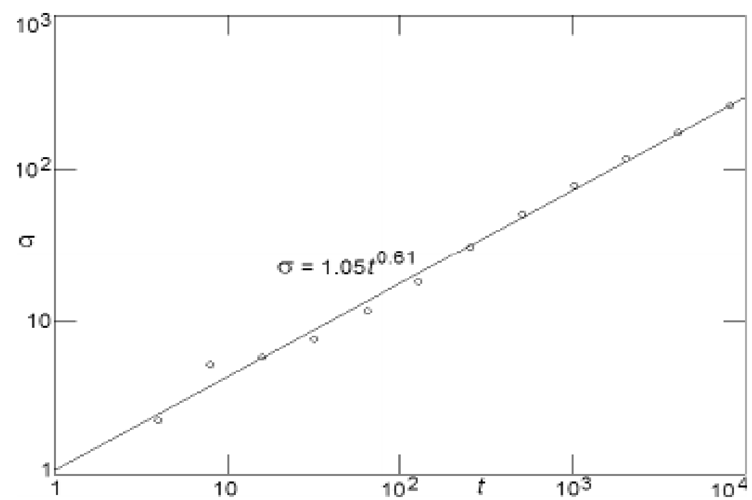


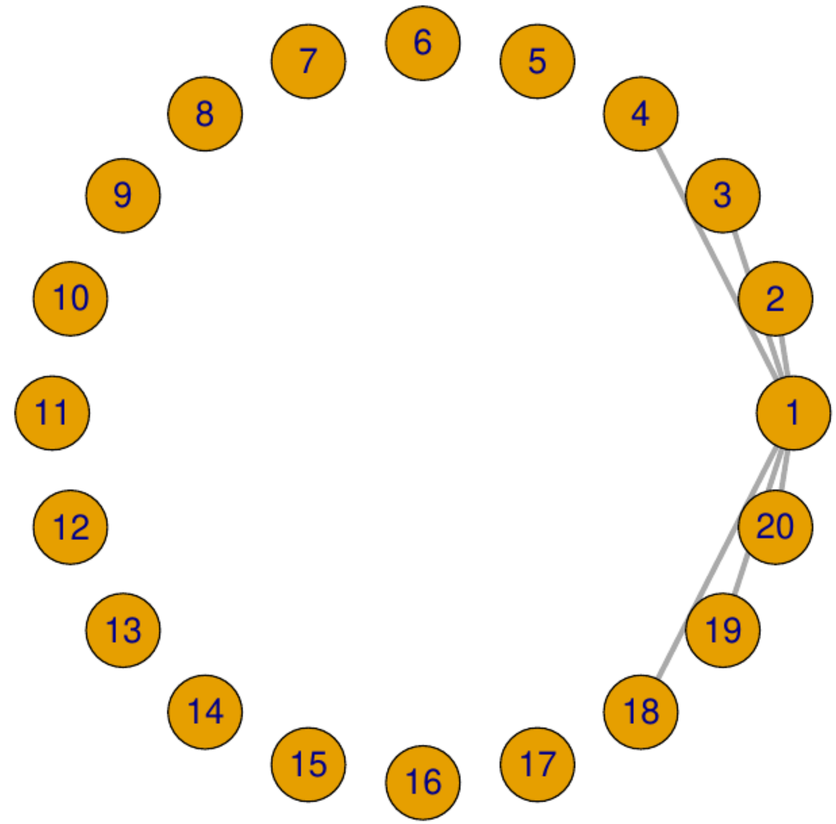
Fig. 15. Standard deviation of 1.5×10^6 trajectories starting near the origin versus time.

“Hyperchaos & Labyrinth Chaos: revisiting Thomas-Rössler systems”
V. Basios , C.G. Antonopoulos, **J. Theor. Biol. 460:153-159, (2019)**

$$\frac{dx_k}{dt} = -b_k x_k + \sin(y_k) + \frac{d}{2P}\sum_{j=k-P}^{k+P}(x_k - x_j),$$

$$\frac{dy_k}{dt} = -b_k y_k + \sin(z_k),$$

$$\frac{dz_k}{dt} = -b_k z_k + \sin(x_k),$$



“Hyperchaos & Labyrinth Chaos: revisiting Thomas-Rössler systems”
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Chimera states 1:

Labyrinth Chaos
&
Complex Periodic Oscillations

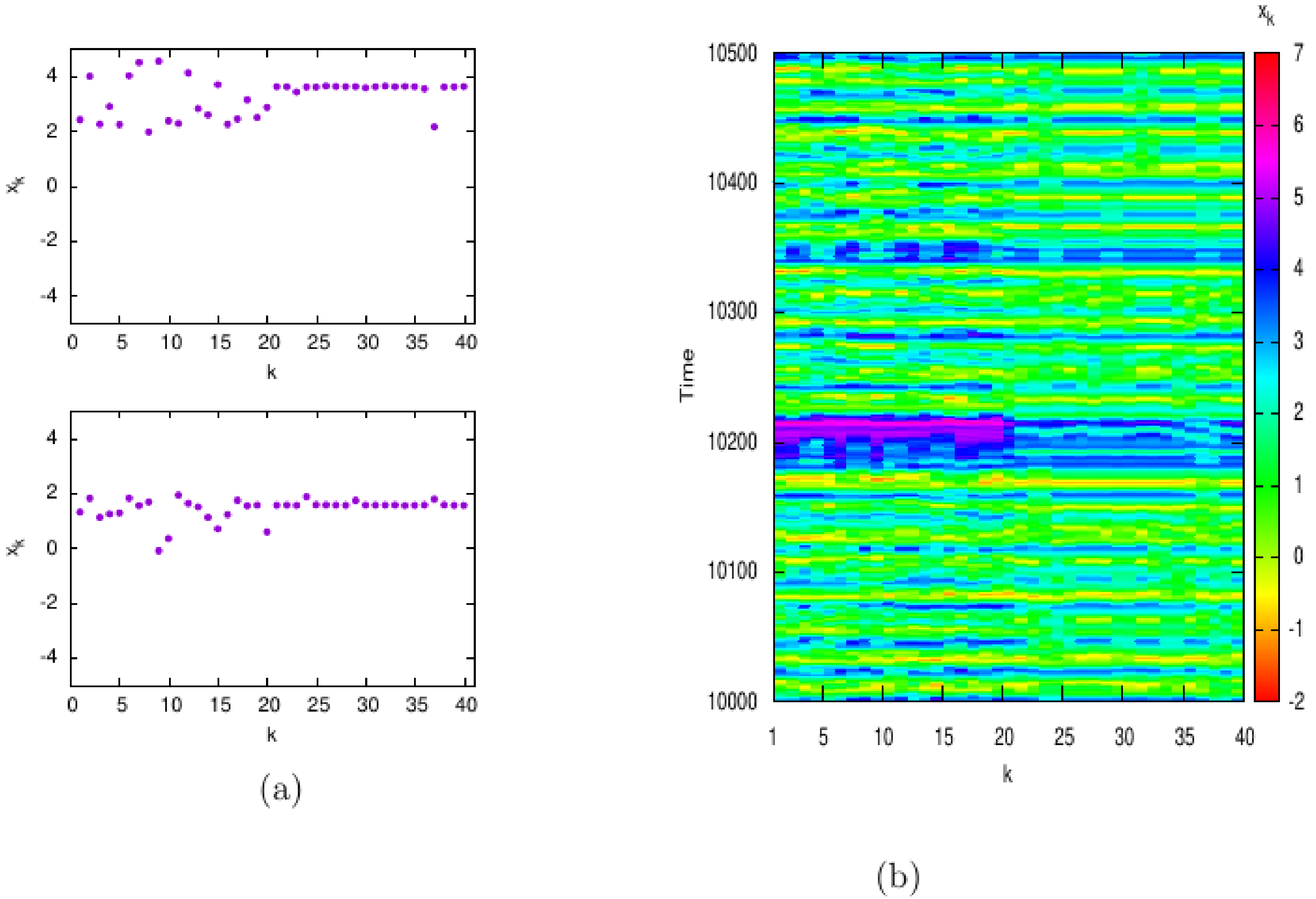
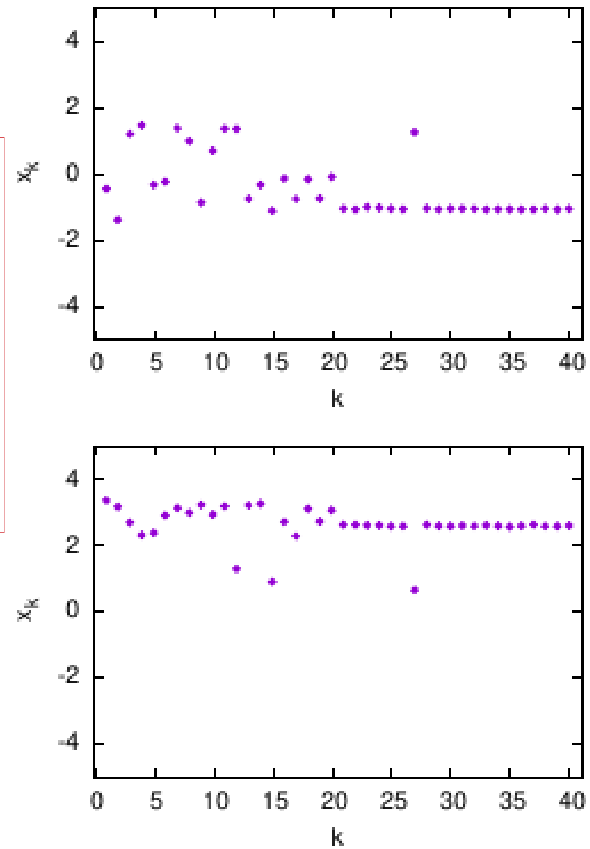


Figure 6: Spatio-temporal phenomena of coherent and incoherent patterns, reminiscent of chimera states in 40 3-dimensional TR linearly coupled systems that exhibit labyrinth chaos and complex periodic oscillations with $b_k = 0$ for $k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.19$ for $k = 21, \ldots, 40$ (complex periodic oscillations). The upper plot in panel (a) is for $t = 10184$ and the lower for $t = 10371$. Panel (b) shows the spatio-temporal patterns between $t = 10000$ and $t = 10500$. Note that in these plots, $d = 0.6$.

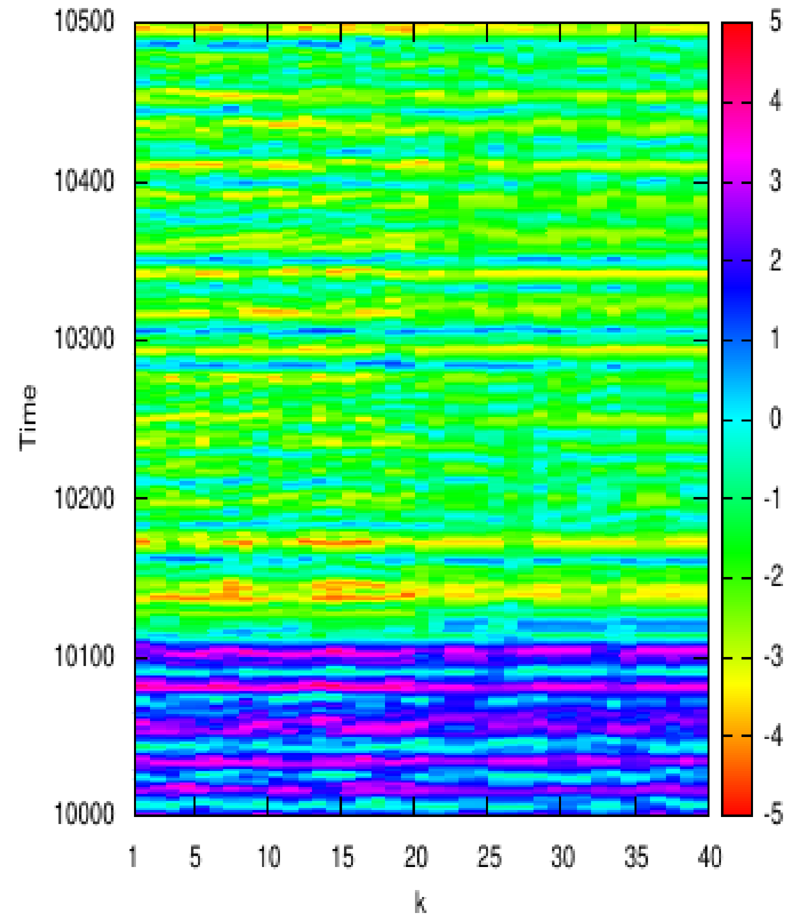
“Hyperchaos & Labyrinth Chaos: revisiting Thomas-Rössler systems”
V. Basios , C.G. Antonopoulos, **J. Theor. Biol. 460:153-159, (2019)**

Chimera states 2:

Labyrinth Chaos
&
Hyper-chaos!



(a)



(b)

Figure 5: Spatio-temporal phenomena of coherent and incoherent patterns, reminiscent of chimera states in 40 3-dimensional TR linearly coupled systems that exhibit labyrinth chaos and hyperchaos with $b_k = 0$ for $k = 1, \ldots, 20$ (labyrinth chaos) and $b_k = 0.18$ for $k = 21, \ldots, 40$ (hyperchaos). The upper plot in panel (a) is for $t = 14462$ and the lower for $t = 14515$. Panel (b) shows the spatio-temporal patterns between $t = 10000$ and $t = 10500$. Note that in these plots, $d = 0.6$.

In search of a Hamiltonian

$$\dot{X} = f(X)$$

$$\nabla H^T f\ (X) = 0$$

$$\dot{X} = J(X)\nabla H$$

“dot” denotes the time-derivative,
X an n-dimensional vector field
f a smooth function from $\mathbf{R}_n$ to $\mathbf{R}_n$
∇**H** ^{T} being the transposed of the vector ∇**H**
H(X) Hamiltonian, ‘energy’
J(X) is a skew symmetric matrix
satisfying the Jacobi’s closure condition

e.g. a Hamiltonian for the Lotka-Volterra System

$$\dot{x} = x(a - by)$$

$$\dot{y} = y(-c + dx)$$

$$H(x, y) = c \ln x + a \ln y - dx - by$$

Manfred Plank: "Hamiltonian structures for LV equations"
J. Math. Phys. 36 (7), July 1995.

Labyrinth Chaos Hamiltonian?

$$\dot{X} = f(X)$$

→

$$\frac{dx}{dt} = -bx + \sin(y)$$
$$\frac{dy}{dt} = -by + \sin(z)$$
$$\frac{dz}{dt} = -bz + \sin(x)$$

$$f_c(X) = \begin{pmatrix} \sin(y) \\ \sin(z) \\ \sin(x) \end{pmatrix} \quad ; \quad f_d(X) = \begin{pmatrix} -bx \\ -by \\ -bz \end{pmatrix}.$$

Strategy: “Reductio ad absurdum”

Assume there is an $H(x,y,z)$,

then prove that $H(x,y,z)$

is either zero or impossible!

$$\nabla H^T f_c(X) = 0$$

$$\frac{\partial H}{\partial x}\sin(y) + \frac{\partial H}{\partial y}\sin(z) + \frac{\partial H}{\partial z}\sin(x) = 0$$

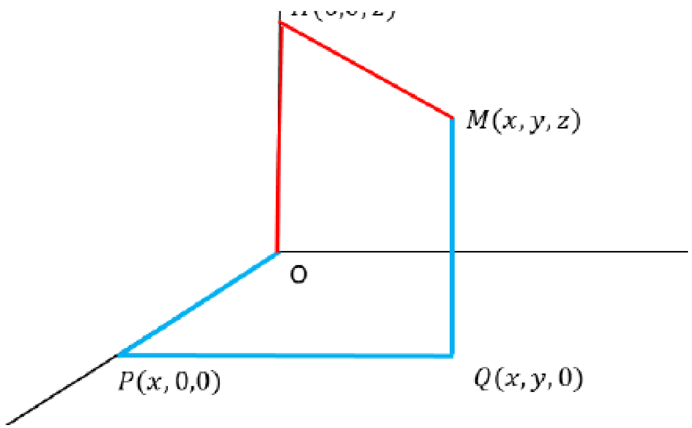
Mechanics' View: Forces and Potentials

The system being conservative, the potential as the opposite of the path-integral of the force is path-independent. … IT IS NOT

$$\frac{d^2x}{dt^2} = \sin(z)\cos(y)$$

$$\frac{d^2y}{dt^2} = \sin(x)\cos(z)$$

$$\frac{d^2z}{dt^2} = \sin(y)\cos(x)$$



$$U(x, y, z) = -y\sin(x) - z\sin(y)\cos(z).$$

$$\left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

Yet, there is a Vector Potential (for b=0) … easy

It worth remarking that in spite the fact that the system (3) is not Hamiltonian, it does have a vector potential. Indeed, it is easy to see that $\nabla f_c = 0$. Thus, there exist a field $F(F_1, F_2, F_3)$, called the vector potential [17], such that $\nabla \times F = f_c$ yielding to the following system

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \sin(y) \tag{13a}$$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \sin(z) \tag{13b}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \sin(x) \tag{13c}$$

As we know, vector potential is not unique. To find a simple solution, we can let $F_3 = 0$ and straight forward calculations yields to

$$F(-\cos(z), -z\sin(y) - \cos(x), 0). \tag{14}$$

Remind that for a conservative system, a vector potential is related to the flow of the field vector f_c through the Stokes' theorem.

Local Hamiltonian Structure?

$$\frac{\partial H}{\partial x}\sin(y) + \frac{\partial H}{\partial y}\sin(z) + \frac{\partial H}{\partial z}\sin(x) = 0$$

There is no function H satisfying the above

- Even locally, it is not possible to find such a function.
- To exhibit the local structure we are going to replace the sinus function by its first terms of its Taylor expansion (b = 0) yields

$$\frac{dx}{dt} = y - \frac{1}{6}y^3$$
$$\frac{dy}{dt} = z - \frac{1}{6}z^3$$
$$\frac{dz}{dt} = x - \frac{1}{6}x^3$$

$$\frac{d^2x}{dt^2} = \left(z - \frac{1}{6}z^3\right)\left(1 - \frac{1}{2}y^2\right)$$
$$\frac{d^2y}{dt^2} = \left(x - \frac{1}{6}x^3\right)\left(1 - \frac{1}{2}z^2\right)$$
$$\frac{d^2z}{dt^2} = \left(y - \frac{1}{6}y^3\right)\left(1 - \frac{1}{2}x^2\right)$$

There is a Hamiltonian iff:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$
$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0$$
$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{F} = 0$$

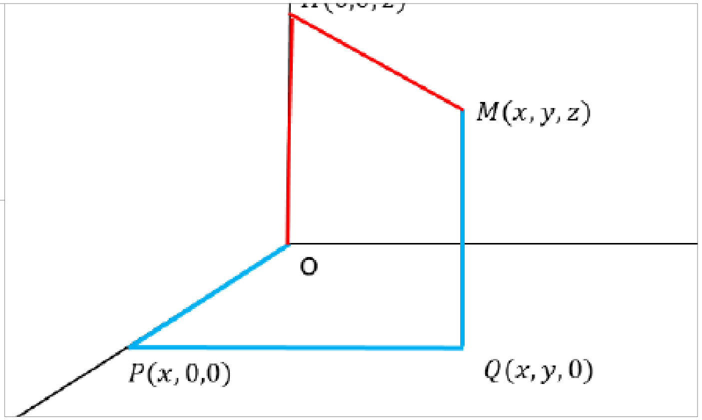
Which yields after simplification

$$\left(1 - \frac{1}{2}x^2\right)\left(1 - \frac{1}{2}y^2\right) + xz\left(1 - \frac{1}{6}x^2\right) = 0$$
$$\left(1 - \frac{1}{2}y^2\right)\left(1 - \frac{1}{2}z^2\right) + xy\left(1 - \frac{1}{6}y^2\right) = 0$$
$$\left(1 - \frac{1}{2}x^2\right)\left(1 - \frac{1}{2}z^2\right) + yz\left(1 - \frac{1}{6}z^2\right) = 0$$

which is obviously not true for any x, y, z.

2nd Way: the Path-Integrals' Independence

$$dU = -\overrightarrow{F}\,.\,\overrightarrow{dr}$$



$$U = -\int_0^z F_{z'}\Big|_{\substack{z'=0\\ y'=0}} dz' - \sqrt{2}\int_0^y F_{y'}\Big|_{\substack{x'=y'\\ z'=z}} dy'$$

$$U = -\int_0^x F_{x'}\Big|_{\substack{y'=0\\ z'=0}} dx' - \int_0^y F_{y'}\Big|_{\substack{x'=x\\ z'=0}} dy' - \int_0^z F_{z'}\Big|_{\substack{x'=x\\ y'=y}} dz'$$

$$U = -\frac{\sqrt{2}}{2} y^2 \left(1 - \frac{1}{2} z^2\right) \left(y - \frac{1}{6} y^2\right)$$

$$\oint \overrightarrow{F} . \overrightarrow{dr} \neq 0$$

$$U = -y \left(x - \frac{1}{6} x^3\right) - z \left(y - \frac{1}{6} y^3\right) \left(1 - \frac{1}{2} x^2\right)$$

Still … exists a Vector Potential ...

$$\vec{\nabla} \times \vec{A} = f_c$$

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = y - \frac{1}{6}y^3$$
$$\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = z - \frac{1}{6}z^3$$
$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = x - \frac{1}{6}x^3$$

$$\vec{A}\left(\frac{1}{2}z^2(1 - \frac{1}{6}z^2)\ ,\ -z(y + \frac{1}{6}y^3)\ ,\ 0\right)$$

カモノハス—急上昇—

一幅、絹本彩色、35.4×75.8 cm、2014-15年

Conclusions & Outlook

Elegant, conservative, path-dependent, non-Hamiltonian, chaos without any attractors.

Chimera-like states ability and the vector potential we shall seek not-energy driven phase-coupling.

Related? “active information transfer” and vector potential (B. Hiley)

Related? “ABC” turbulence model(s). Other instances of similar systems? (LLV variants?)

Its Symmetries and Symbolic Dynamics