Further and More on Equilibria of 3D Autonomous Chaotic Systems

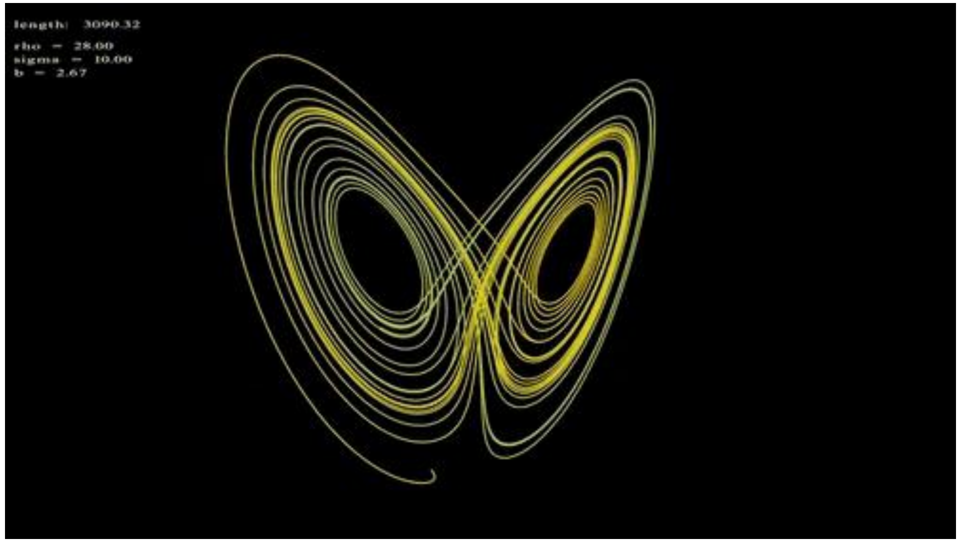
GuanRong Chen

CityU
香港城市大學
City University of Hong Kong

Lorenz System

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases}$$

$$a = 10, b = 8/3, c = 28$$



E. N. Lorenz, “Deterministic non-periodic flow,”
J. Atmos. Sci., 20: 130-141, 1963.

Main Characteristics

“Simple”

Autonomous

3D Quadratic (smooth)

3 Equilibria (two saddles)

Hyperbolic (Jacobian eigenvalues (- , 0, +))

After all, it is chaotic

3D Autonomous Chaotic Systems

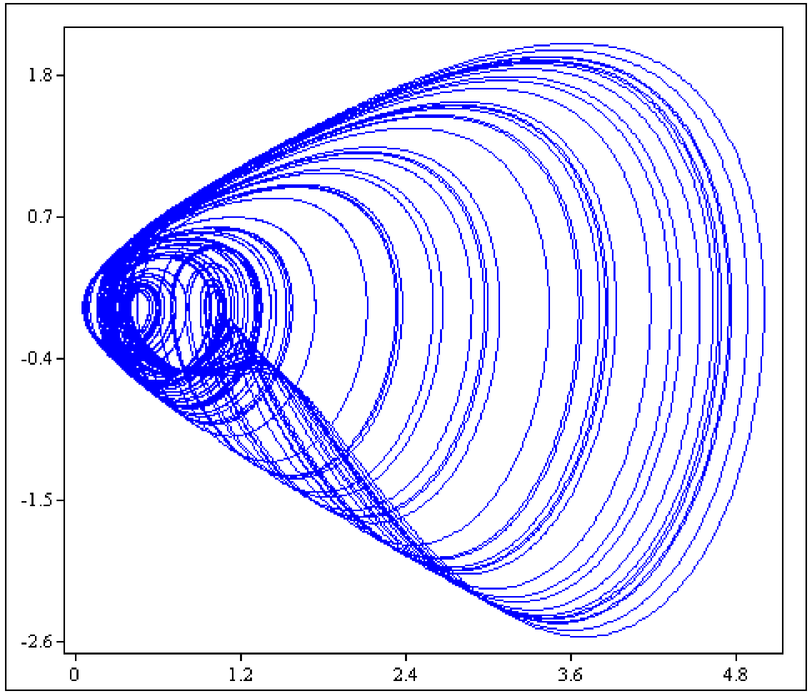
State of the Art: with

- ✓ No equilibrium points
- ✓ One equilibrium point
- ✓ Two equilibrium points
- ✓ Three equilibrium points
- ✓ Any number of equilibrium points
- ✓ Infinitely many equilibrium points

Chaotic system with no equilibrium points

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -y + 3y^2 - x^2 - xz - a. \end{cases}$$

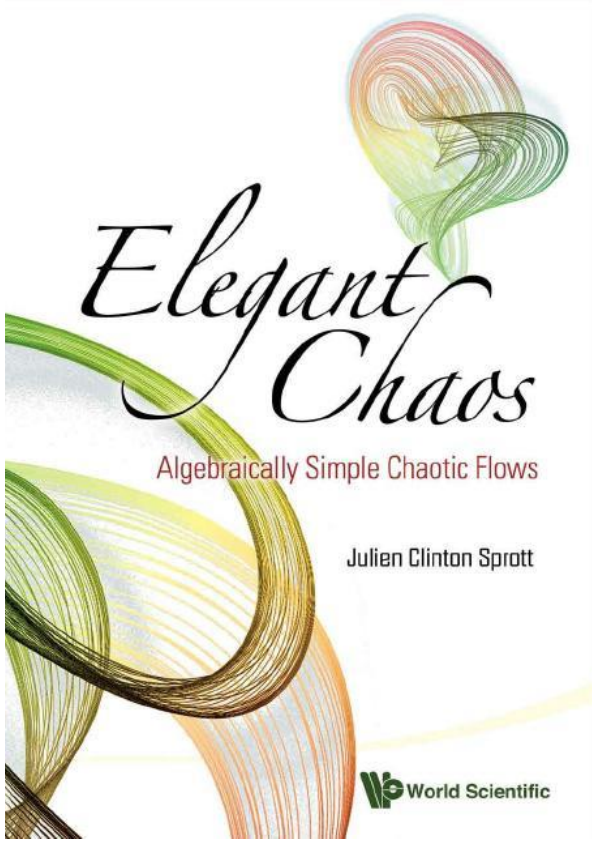
$a > 0$ no equilibrium



X Wang and G Chen: Constructing a chaotic system with any number of equilibria, Nonl Dynam 2013

More: Chaotic systems with no equilibrium (1/2)

Model	Equations	a	LE_s	D_{KY}	(x_0, y_0, z_0)
NE_1	$\dot{x} = y$ $\dot{y} = -x - zy$ $\dot{z} = y^2 - a$	1.0	0.0138, 0, -0.0138	3.0000	(0, 5, 0)
NE_2	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = 2y^2 + xz - a$	0.35	0.0776, 0, -1.5008	2.0517	(0, 0.4, 1)
NE_3	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y + 0.1x_2 + 1.1xz + a$	1.0	0.0522, 0, -2.6585	2.0196	(1, 1, - 1)
NE_4	$\dot{x} = -0.1y + a$ $\dot{y} = x + z$ $\dot{z} = xz - 3y$	1.0	0.0235, 0, -8.480	2.0028	(-8.2, 0, -5)
NE_5	$\dot{x} = 2y$ $\dot{y} = -2x - z$ $\dot{z} = -y^2 + z^2 + a$	2.0	0.0168, 0, -0.3622	2.0465	(0.98, 1.8, -0.7)
NE_6	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y - xz - yz - a$	0.75	0.0280, 0, -3.4341	2.0082	(0, 3, -0.1)
NE_7	$\dot{x} = y$ $\dot{y} = -x + z$ $\dot{z} = -0.8x^2 + z^2 + a$	2.0	0.0252, 0, -6.8524	2.0037	(0, 2.3, 0)
NE_8	$\dot{x} = y$ $\dot{y} = -x - yz$ $\dot{z} = xy + 0.5x^2 - a$	1.3	0.0314, 0, -10.2108	2.0031	(0, 0.1, 0)



S. Jafaria, J.C. Sprott, S. M. R. H. Golpayegani, Elementary quadratic chaotic flows with no equilibria, Phys. Lett. A 377, 699-702 (2013)

Chaotic systems with no equilibrium (2/2)

NE_9	$\dot{x}=y$ $\dot{y}=-x-yz$ $\dot{z}=-xz+7x^2-a$	0.55	0.0504, 0, -0.3264	2.1544	(0.5, 0, 0)
NE_{10}	$\dot{x}=z$ $\dot{y}=z-y$ $\dot{z}=-0.9y-xy+xz+a$	0.6	0.0061, 0, -1.3002	2.0047	(1, 0.7, 0.8)
NE_{11}	$\dot{x}=y$ $\dot{y}=-x+z$ $\dot{z}=z-2xy-1.8xz-a$	1.0	0.0706, 0, -0.6456	2.1094	(0, 1.6, 3)
NE_{12}	$\dot{x}=z$ $\dot{y}=x-y$ $\dot{z}=-4x2+8xy+yz+a$	0.1	0.0654, 0, -2.0398	2.0321	(0.5, 0, -1)
NE_{13}	$\dot{x}=-y$ $\dot{y}=x+z$ $\dot{z}=xy+xz+0.2yz-a$	0.4	0.1028, 0, -2.1282	2.0483	(2.5, 0, 0)
NE_{14}	$\dot{x}=y$ $\dot{y}=z$ $\dot{z}=x^2-y^2+2xz+yz+a$	1.0	0.0532, 0, -11.8580	2.0045	(1, 0, -4)
NE_{15}	$\dot{x}=y$ $\dot{y}=z$ $\dot{z}=x^2-y^2+xy+0.4xz+a$	1.0	0.1101, 0, -1.3879	2.0793	(0, 1, -4.9)
NE_{16}	$\dot{x}=-0.8x-0.5y^2+xz+a$ $\dot{y}=-0.8y-0.5z^2+yx+a$ $\dot{z}=-0.8z-0.5x^2+zy+a$	1.0	0.0607, 0, -0.1883	2.3224	(0, 1, -1)
NE_{17}	$\dot{x}=-y-z^2+2.3xy+a$ $\dot{y}=-z-x^2+2.3yz+a$ $\dot{z}=-x-y^2+2.3zx+a$	2.0	0.2257, 0, -1.7477	2.1292	(1, -1, 0)

S. Jafaria, J.C. Sprott, S. M. R. H. Golpayegani, Elementary quadratic chaotic flows with no equilibria, Phys. Lett. A 377, 699-702 (2013)

Chaotic system with one stable equilibrium

$$\begin{cases} \dot{x} = yz + a \\ \dot{y} = x^2 - y \\ \dot{z} = 1 - 4x \end{cases}$$

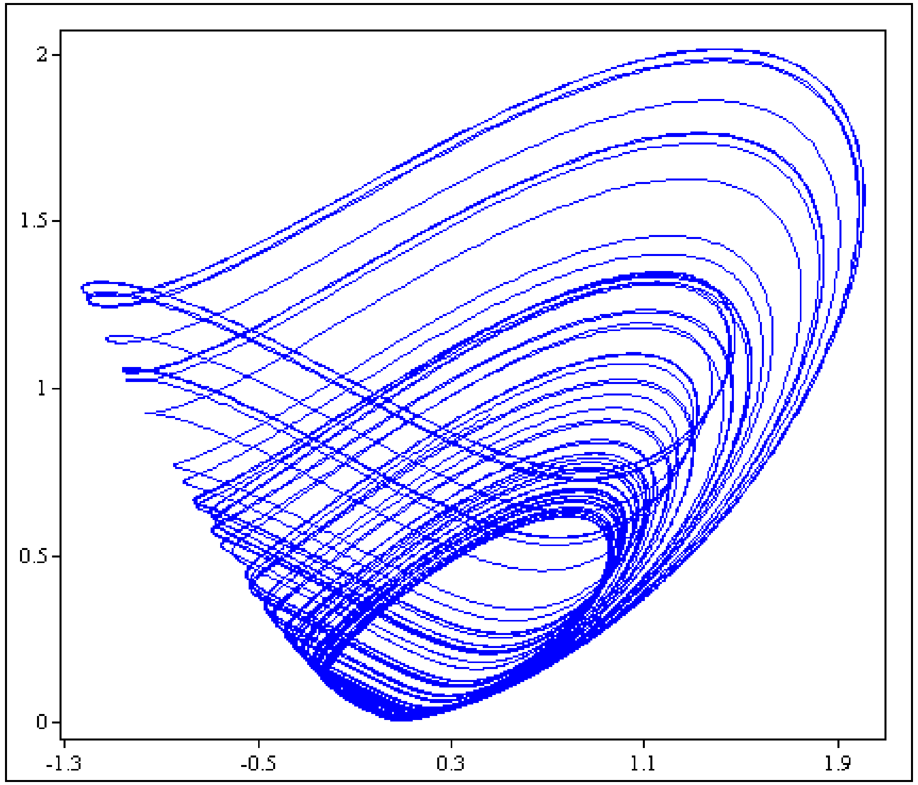
$$\left(\frac{1}{4}, \frac{1}{16}, -16a\right)$$

a = - 0.005

(- 1.03140, 0.01570 ± 0.49208i)

a = + 0.05

(- 0.60746, - 0.19627 ± 0.61076i)



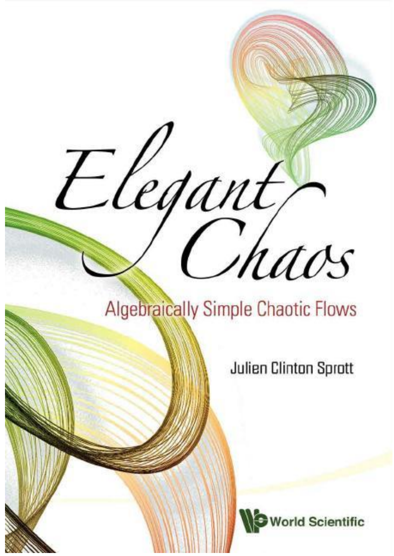
X Wang and G Chen: A chaotic system with only one stable equilibrium, Comm in Nonl Sci and Numer Simul, 2012

More: chaotic systems with one equilibrium

Table 1. 23 simple chaotic flows with one stable equilibrium.

Model	Equations	Equilibrium	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
SE_1	$\dot{x} = y$	0	−1.9548	0.0377		4
	$\dot{y} = z$	0	−0.0226	0	2.0185	−2
	$\dot{z} = -x - 0.6y - 2z + z^2 - 0.4xy$	0	±0.7149*i*	−2.0377		0
SE_2	$\dot{x} = y$	0	−0.5103	0.0804		−1
	$\dot{y} = z$	0	−0.0198	0	2.1644	0
	$\dot{z} = -0.5x - y - 0.55z - 1.2z^2 - xz - yz$	0	±0.9896*i*	−0.4889		1
SE_3	$\dot{x} = y$	0	−3.9641	0.0711		−2
	$\dot{y} = z$	0	−0.0179	0	2.0175	0
	$\dot{z} = -3.4x - y - 4z + y^2 + xy$	0	±0.9259*i*	−4.0711		2.4
SE_4	$\dot{x} = y$	−1	−1.6942	0.0434		0.5
	$\dot{y} = z$	0	−0.0029	0	2.0249	1
	$\dot{z} = -x - 1.7z + y^2 + 0.6xy - 1$	0	±0.7683*i*	−1.7434		0
SE_5	$\dot{x} = y$	−2.7	−0.9600	0.0136		−6.1
	$\dot{y} = z$	0	−0.0200	0	2.0134	1
	$\dot{z} = -x - z - z^2 + 0.4xy - 2.7$	0	±1.0204*i*	−1.0136		1
SE_6	$\dot{x} = y$	−1	−1.0526	0.0638		−2.2
	$\dot{y} = z$	0	−0.0237	0	2.0600	0.6
	$\dot{z} = -x - 2.9z^2 + xy + 1.1xz - 1$	0	±0.9744*i*	−1.0638		0
SE_7	$\dot{x} = y$	0	−2.0000	0.0360		1
	$\dot{y} = -x + yz$	0	−0.2500	0	2.0014	−0.7
	$\dot{z} = -2z - 8xy + xz - 1$	−0.5	±0.9682*i*	−25.6798		0
SE_8	$\dot{x} = y$	0	−1.0000	0.1412		0
	$\dot{y} = -x + yz$	0	−0.0500	0	2.1034	0.9
	$\dot{z} = -z - 0.7x^2 + y^2 - 0.1$	−0.1	±0.9987*i*	−1.3649		0

Totally, **23** →



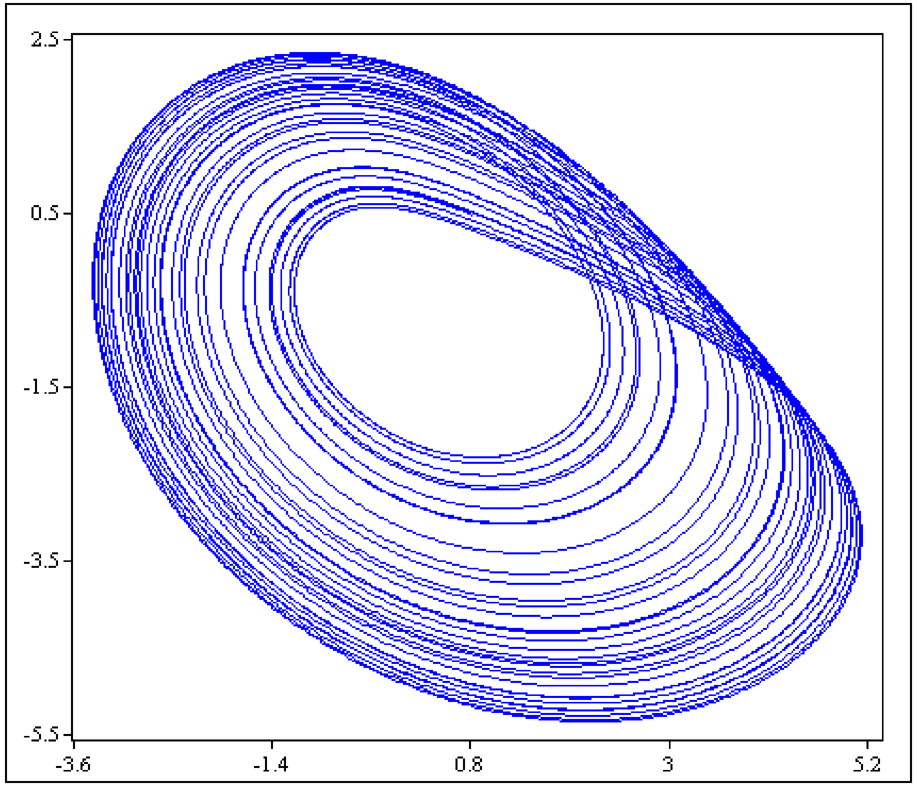
M. Molaie, S. Jafari, J. C. Sprott, and S. M. R. H. Golpayegani, Simple chaotic flows with one stable equilibrium, Int. J. Bifur. Chaos, 23(11), 1350188, 2013.

Chaotic systems with two equilibrium points

Rössler system:

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

$$a = b = 0.2, c = 5.7$$



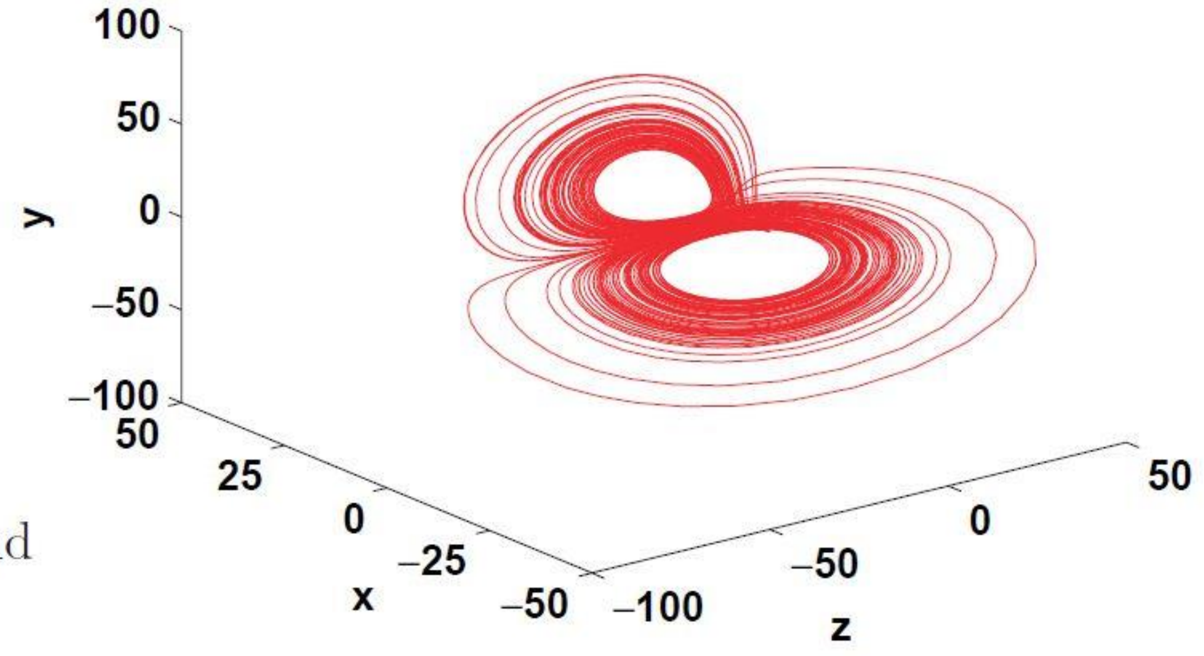
2 Stable Foci

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -cy - xz \\ \dot{z} = -b + xy, \end{cases}$$

a = 10, b = 100, c = 11.2

$$E_1 : (x_0, y_0, z_0) = (\sqrt{b}, \sqrt{b}, -c) \quad \text{and}$$
$$E_2 : (-x_0, -y_0, z_0) = (-\sqrt{b}, -\sqrt{b}, -c)$$

$$\lambda_1 = -20.9778$$
$$\lambda_{2,3} = -0.1111 \pm 9.7635i$$



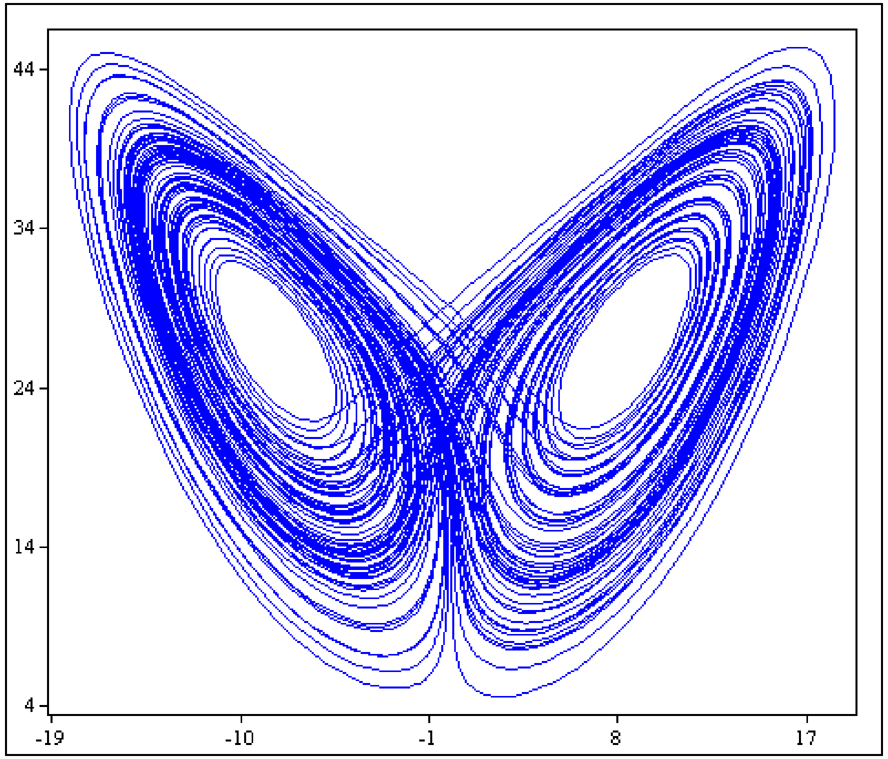
Q. Yang, Z. Wei and G. Chen, IJBC (2010)

Chaotic systems with three equilibrium points

Lorenz system:

$$\dot{x} = a(y - x)$$
$$\dot{y} = x(c - z) - y$$
$$\dot{z} = xy - bz$$

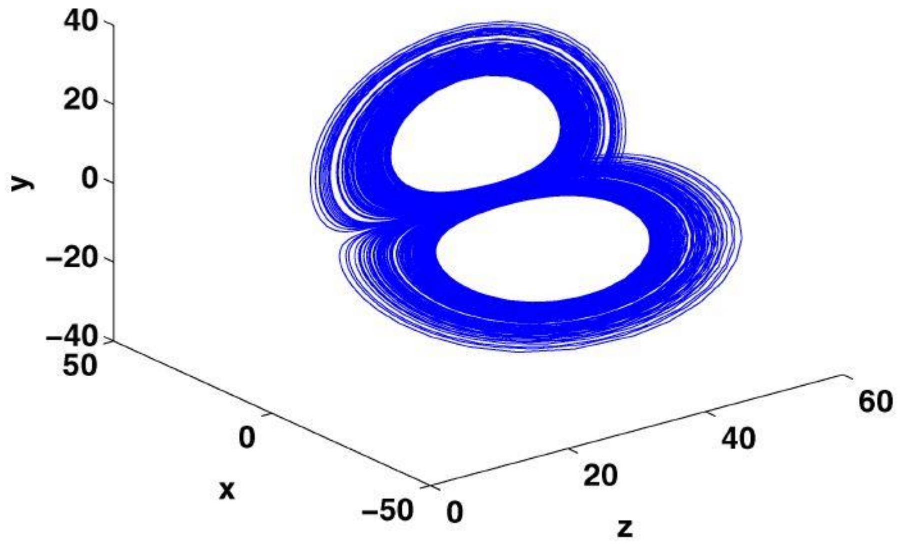
$$a = 10, b = \frac{8}{3}, c = 28$$



1 Saddle + 2 Stable Foci

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz \\ \dot{z} = -bz + xy. \end{cases}$$

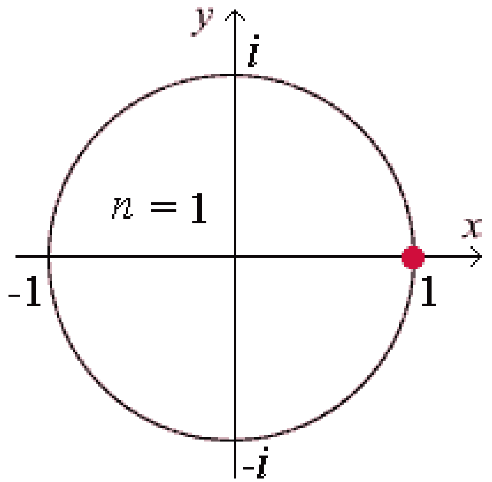
$a = c = 35, b = 3$



Equilibria	Eigenvalues
$(0, 0, 0)$	$-3, -\frac{35(\sqrt{5}+1)}{2}, \frac{35(\sqrt{5}-1)}{2}$
$(\pm\sqrt{105}, \pm\sqrt{105}, 35)$	$-37.6122, -0.1939 \pm 13.9778i$

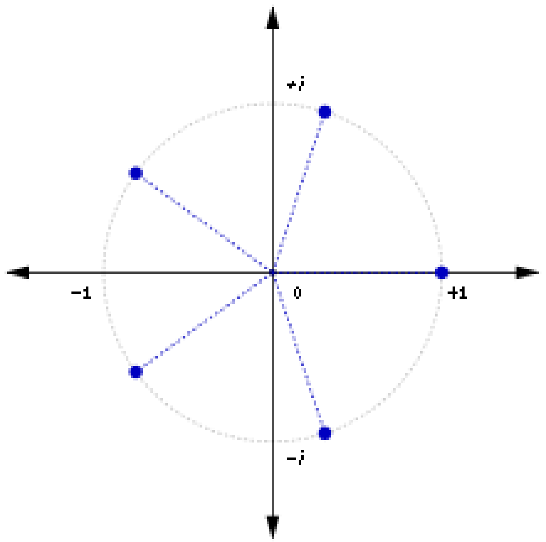
Q. Yang and G. Chen, IJBC (2008)

Chaotic systems with any number of equilibrium points



Idea: Transform

$$W = Z^n$$



W plane

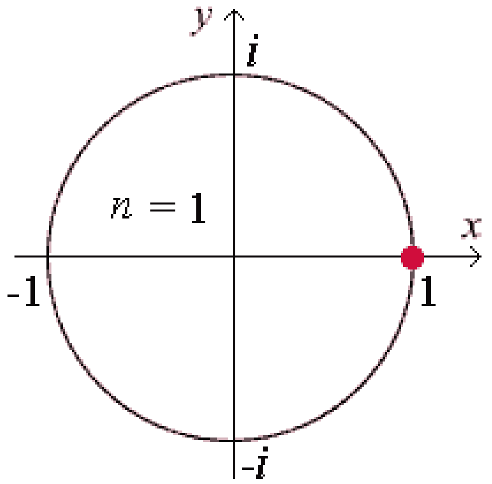
Z plane

$$W = (u,v) = u + vi$$

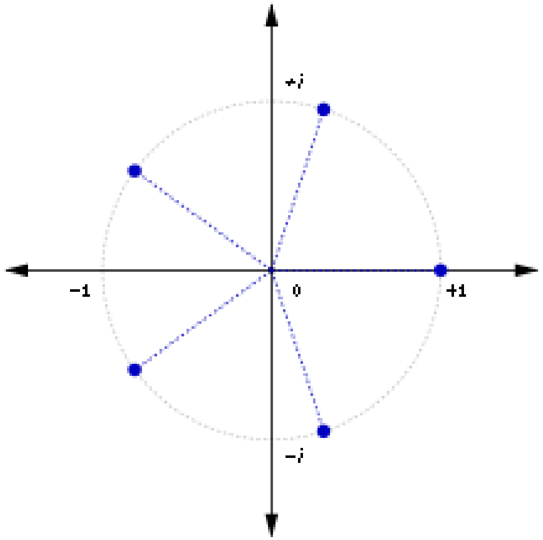
$$Z = (x,y) = x + yi$$

X Wang and G Chen: Constructing a chaotic system with any number of equilibria, Nonl Dynam 2013

Symmetry



$$W = Z^n$$



W plane

$$W = (u,v) = u + vi$$

original system:

$$(u,v,w)$$

Z plane

$$Z = (x,y) = x + yi$$

translated system:

$$(x,y,z)$$

Example

$$\mathbb{R}_y(\pi) \quad \begin{cases} u = x^2 - z^2 \\ v = y \\ w = 2xz \end{cases}$$

local diffeomorphism

$$\begin{cases} \dot{u} = vw + a \\ \dot{v} = u^2 - v \\ \dot{w} = 1 - 4u. \end{cases} \longrightarrow \begin{cases} \dot{x} = \frac{1}{2}\frac{z+2yx^2z+xa-4x^2z+4z^3}{x^2+z^2} \\ \dot{y} = \left(x^2 - z^2\right)^2 - y \\ \dot{z} = -\frac{1}{2}\frac{2yxz^2+za-4xz^2-x+4x^3}{x^2+z^2}. \end{cases}$$

One equilibrium

Two equilibria

Stability of the two equilibrium points

- Two symmetrical equilibrium points (independent of parameter a)

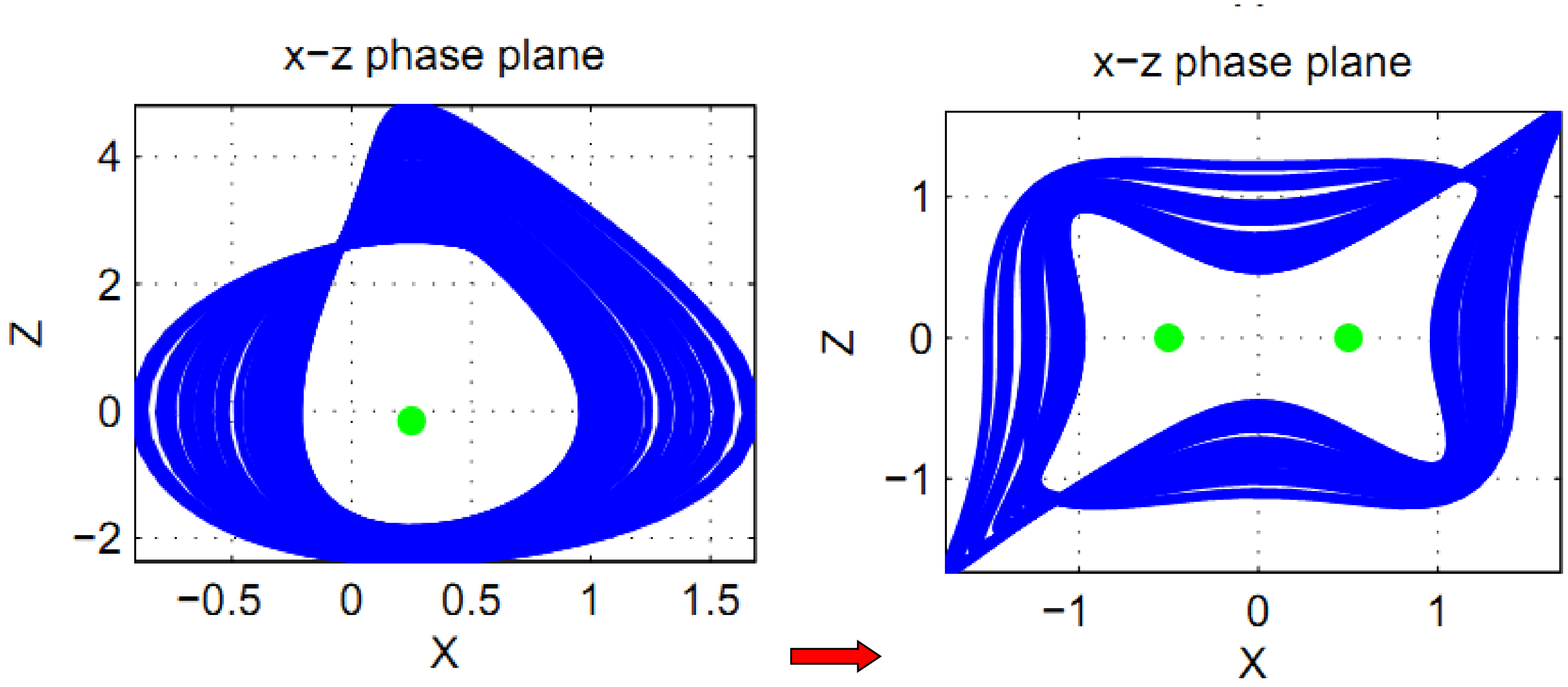
$$P1(\tfrac{1}{2}, \tfrac{1}{16}, 0) \text{ and } P1(-\tfrac{1}{2}, \tfrac{1}{16}, 0)$$

- Eigenvalue of Jacobian:

$$\begin{aligned} \lambda_1 &= -1 < 0, \\ \lambda_2 &= -2a + 0.5i, \\ \lambda_3 &= -2a - 0.5i. \end{aligned}$$

- So, $a > 0$ (stable); $a < 0$ (unstable)

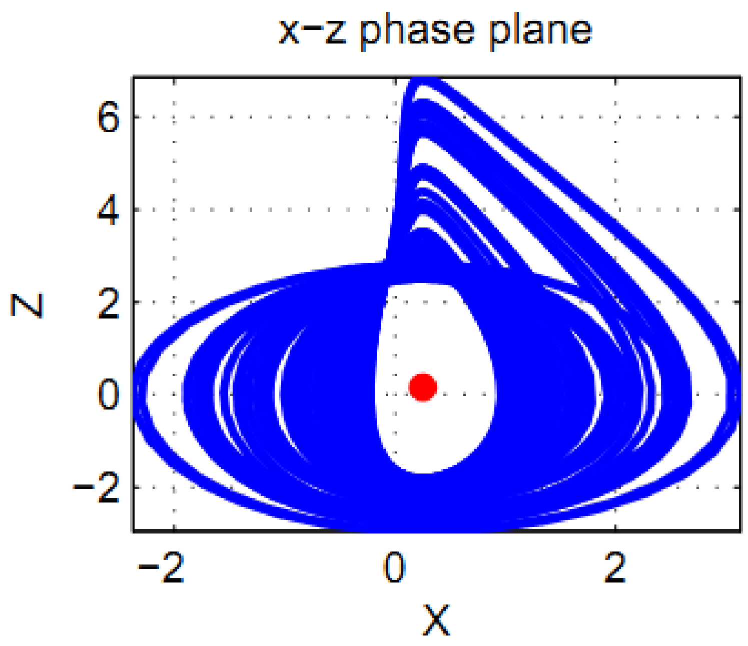
Simulation

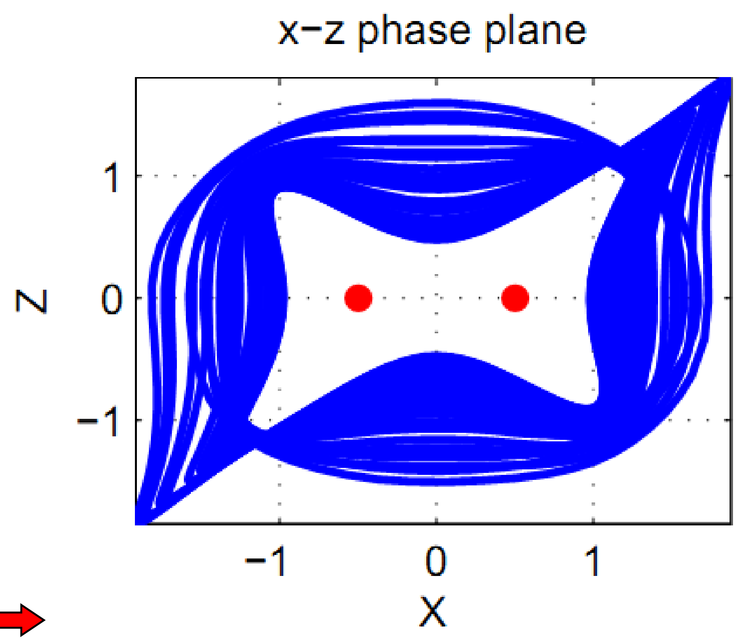


$$a = 0.005 > 0$$

stable equilibrium points

Simulation





$$a = -0.01 < 0$$

unstable equilibrium points

Example

$$\mathbb{R}_y(\tfrac{2}{3}\pi) \quad \begin{cases} u = x^3 - 3xz^2 \\ v = y \\ w = 3x^2z - z^3 \end{cases}$$

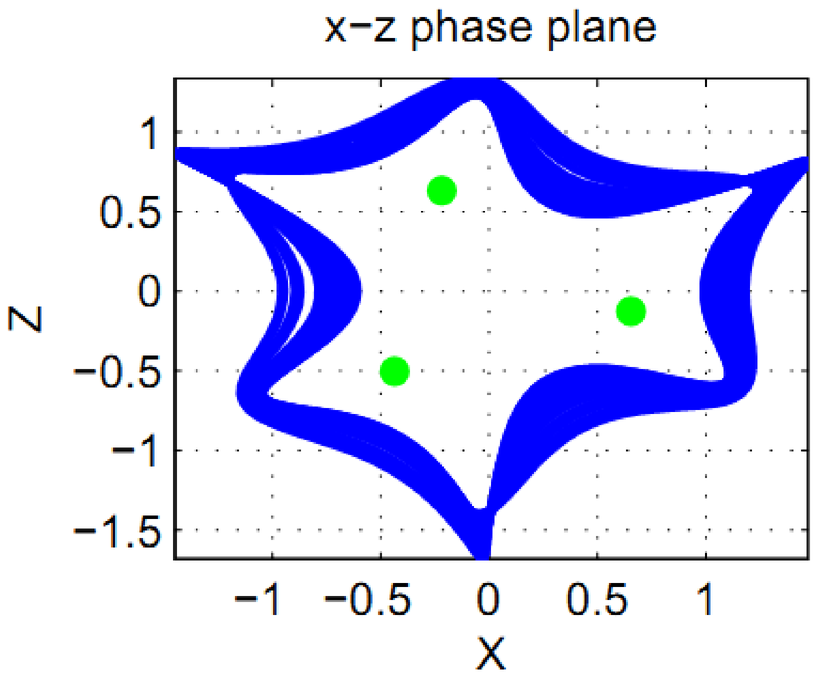
local diffeomorphism

$$\begin{cases} \dot{u} = vw + a \\ \dot{v} = u^2 - v \\ \dot{w} = 1 - 4u. \end{cases}$$

$$\begin{cases} \dot{x} = \frac{1}{3}\,\frac{3\,x^4zy - 4\,x^2z^3y + x^2a - 8\,x^4z + 2\,zx + 24\,x^2z^3 + z^5y - z^2a}{2\,x^2z^2 + x^4 + z^4} \\ \dot{y} = (x^3 - 3xz^2)^2 - y \\ \dot{z} = -\frac{1}{3}\,\frac{6\,z^2x^3y - 2\,z^4xy + 2\,zxa + 4\,x^5 - x^2 - 16\,z^2x^3 + z^2 + 12\,z^4x}{2\,x^2z^2 + x^4 + z^4}. \end{cases}$$

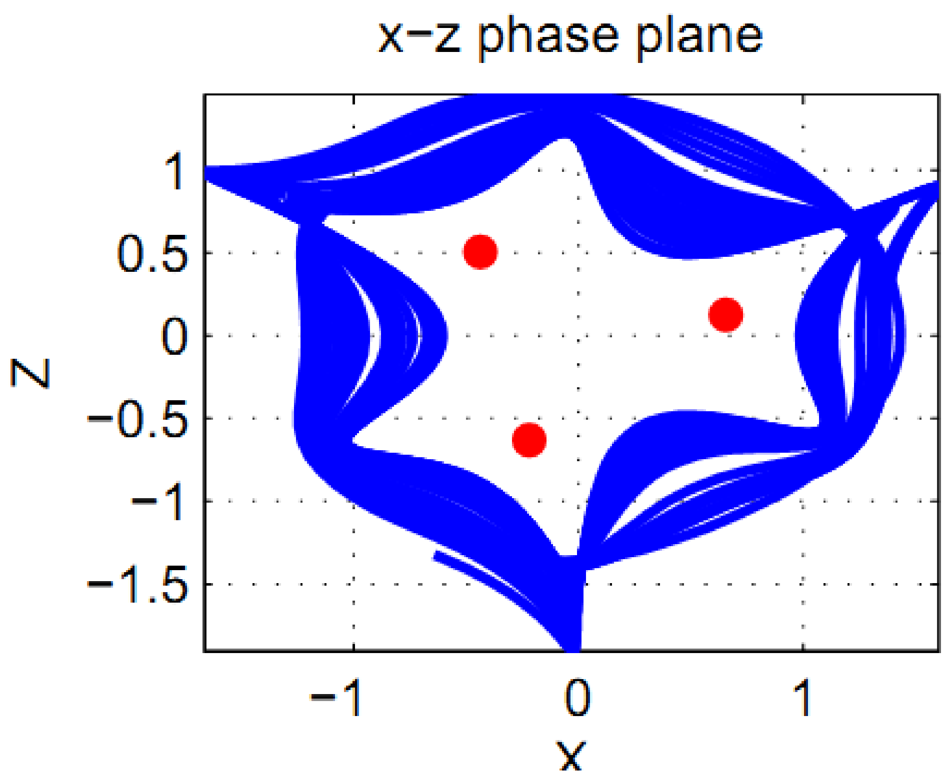
	a	Equilibria	Jacobian eigenvalues
Unstable case	-0.01	$P1 = (0.6550, 0.0625, 0.1258)$ $P2 = (-0.2186, 0.0625, -0.6300)$ $P3 = (-0.4365, 0.0625, 0.5044)$	$-1.0617, 0.0308 \pm 0.4843i$
Stable case	0.01	$P1 = (0.6550, 0.0625, -0.1258)$ $P2 = (-0.2186, 0.0625, 0.6300)$ $P3 = (-0.4365, 0.0625, -0.5044)$	$-0.9334, -0.0333 \pm 0.5165i$

Simulation



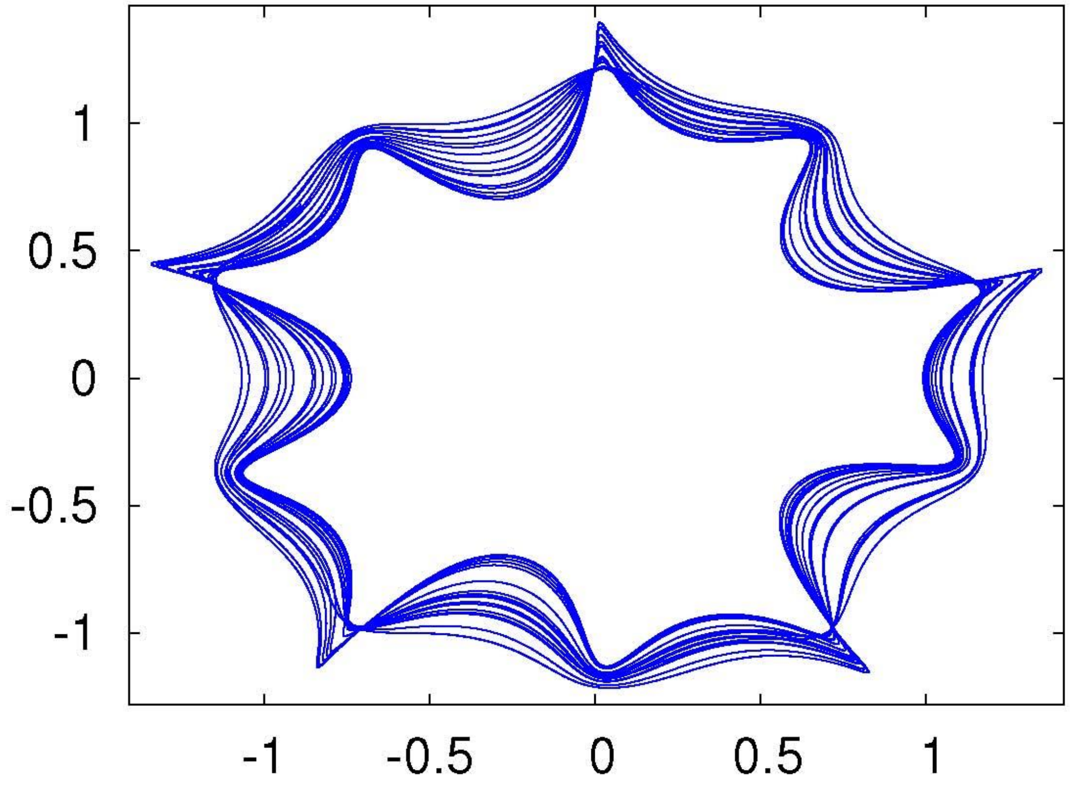
$$a = 0.005 > 0$$ (stable)

Simulation



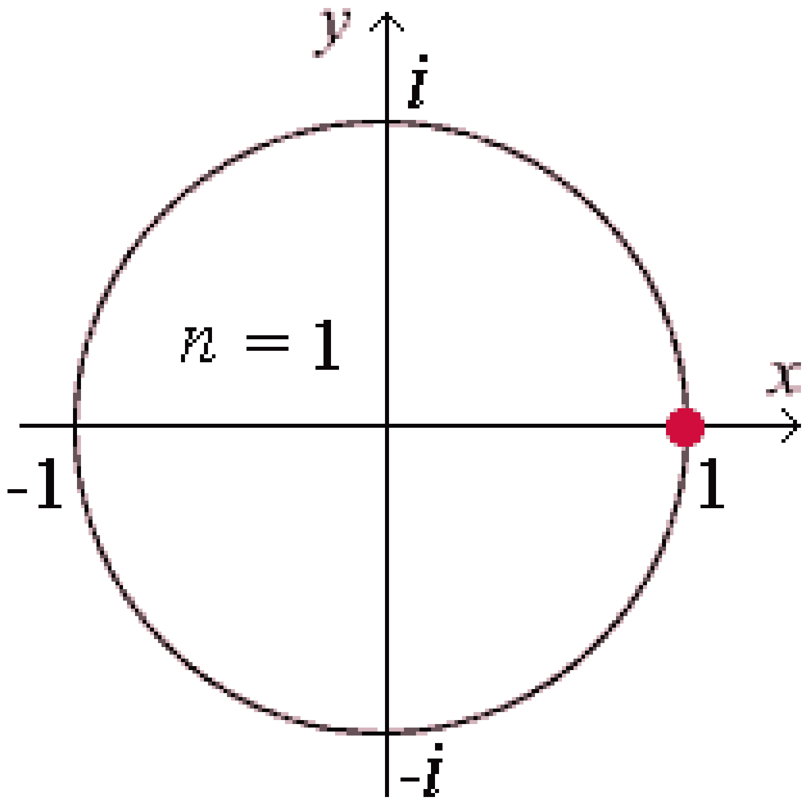
$a = -0.01 < 0$ (unstable)

One more example:



$$R_y\left(\frac{2}{5}\pi\right)$$

Now, theoretically one can create any number of equilibrium points



$$(u+wi)=(x+yi)^n$$

Existence of Chaotic Attractors

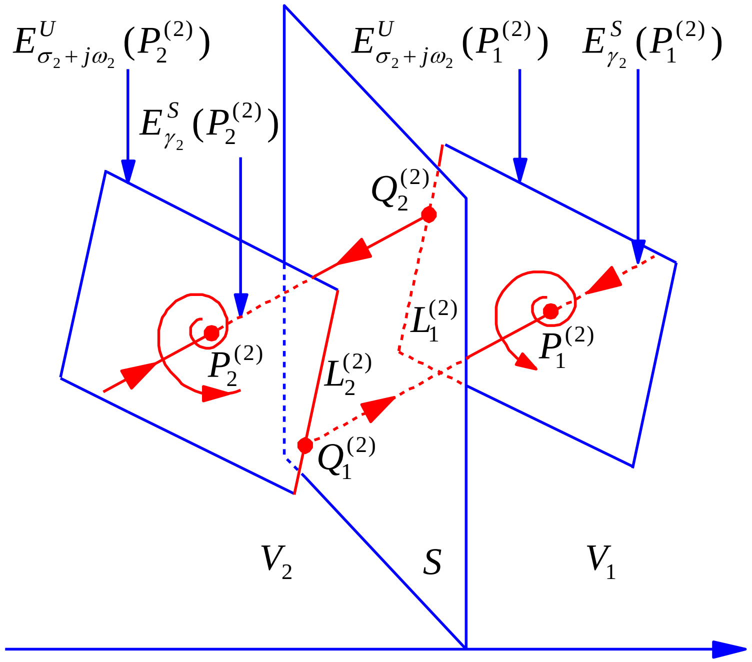
Shilnikov Theorem (1967):

If a 3D autonomous system has two distinct saddle equilibrium points and there exists a heteroclinic orbit connecting them, and if the eigenvalues of the Jacobin of the system at these fixed points are

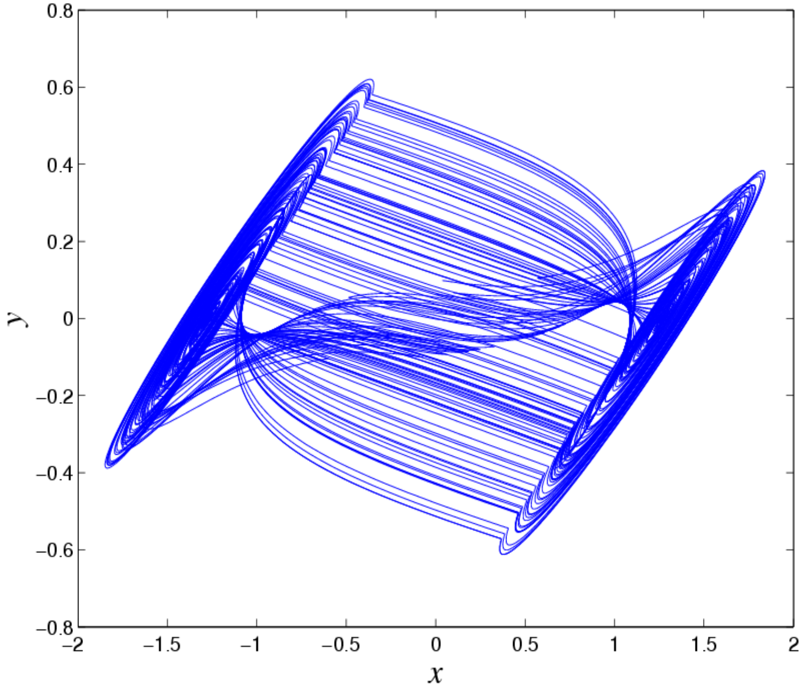
$$\alpha_k, \beta_k \pm j\omega_k \, (k = 1,2)$$

Satisfying $|\alpha_k| > |\beta_k| > 0 \; (k = 1,2)$ and $\beta_1 \beta_2 > 0$ *or* $\omega_1 \omega_2 > 0$

then the system has infinitely many Smale horseshoes and hence has horseshoe chaos.

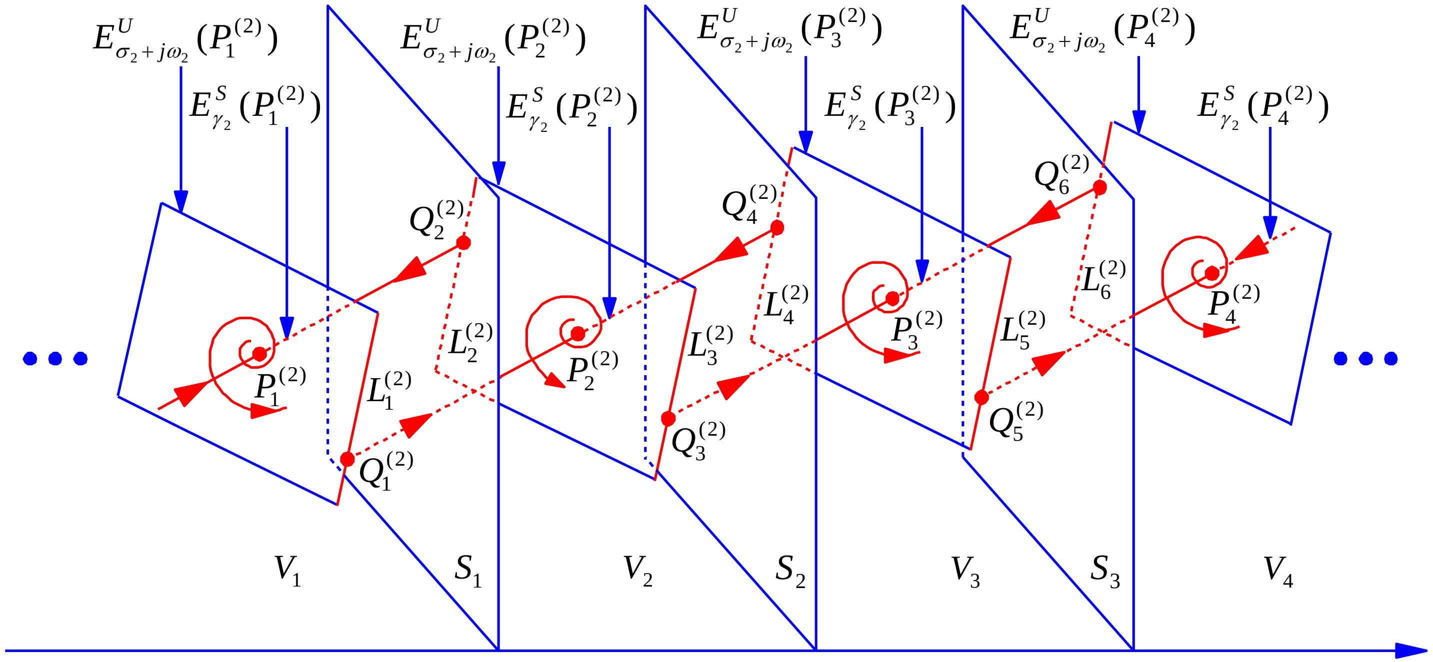


(a) Single heteroclinic orbit

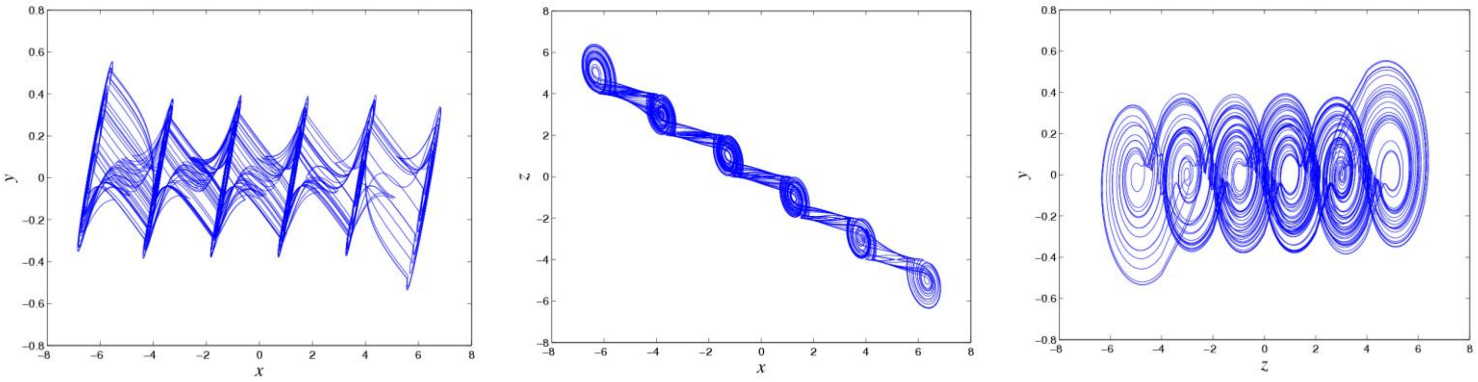


(b) Double-scroll attractor

Extension to Multi-Equilibrium Scenario



(a) Multiple heteroclinic orbit



(b) Multi-scroll attractors

Multi-Equilibrium/Multi-Scroll Attractors in Nature

Thank You !