Granular Waves:

The Dynamical Systems Approach

Ko van der Weele

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Based on the following papers:

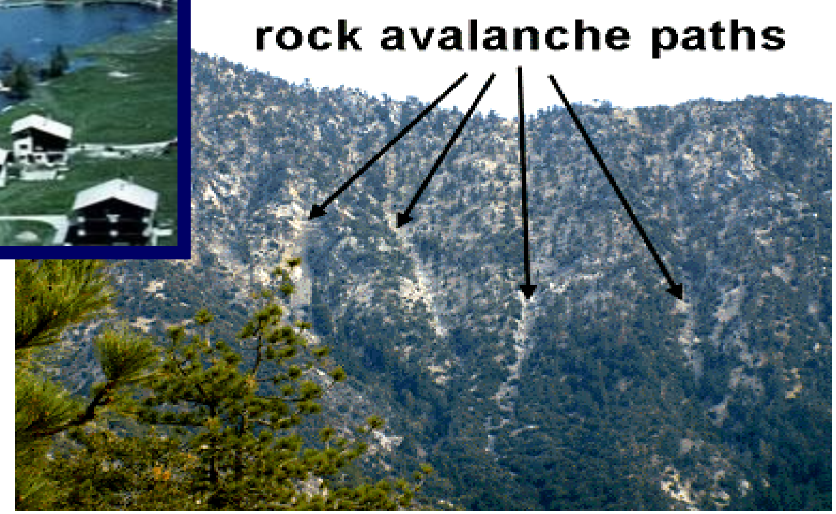
- D. Razis, G. Kanellopoulos, and K. van der Weele,
  A dynamical systems view of granular flow: from monoclinal flood waves to roll waves,
  J. Fluid Mech. **869**, 143-181 (2019).

- D. Razis, G. Kanellopoulos, and K. van der Weele,
  The granular monoclinal wave,
  J. Fluid Mech. **843**, 810-846 (2018).

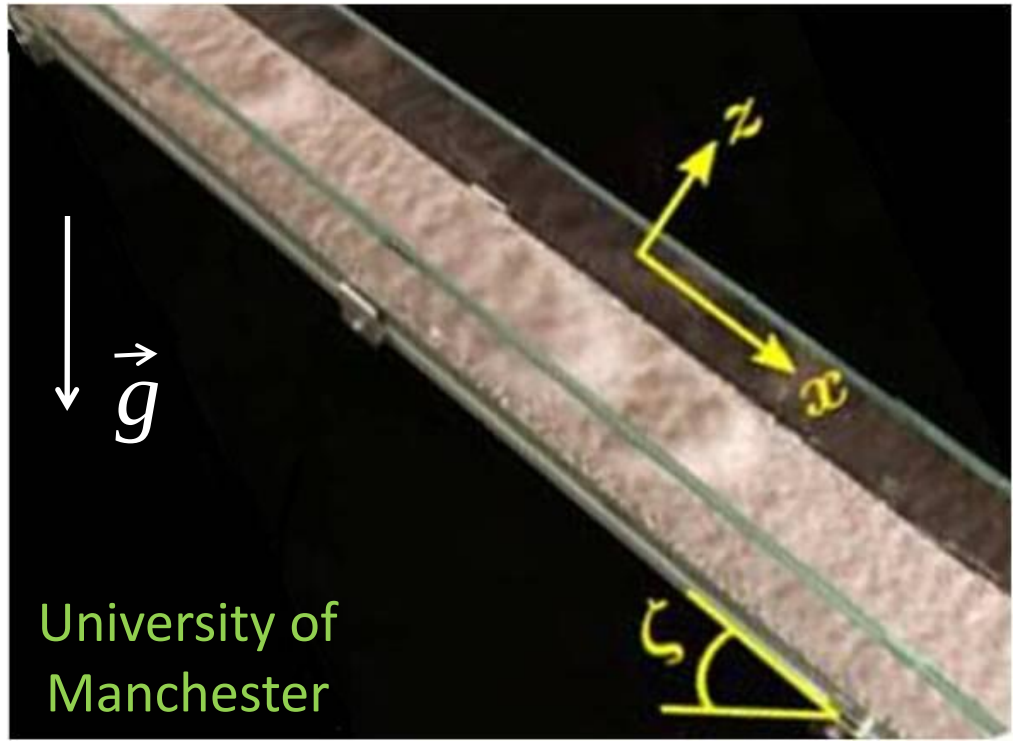
- D. Razis, A.N. Edwards, J.M.N.T. Gray, and K. van der Weele,
  Arrested coarsening of granular roll waves,
  Phys. Fluids **26**, 123305 (2014).

Granular Chute Flow

The "chute" may also be a mountainside:



In the laboratory:

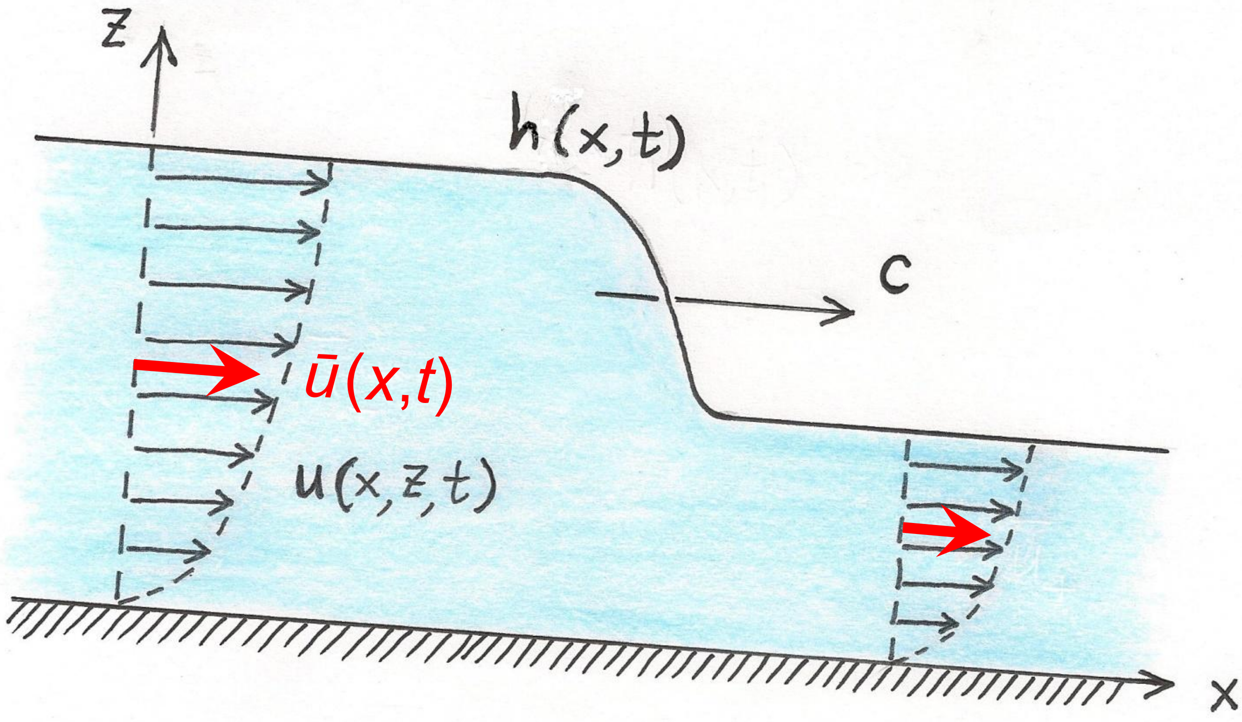


height of the sheet $h(x,t)$

depth-averaged velocity $\bar{u}(x,t)$

Depth-averaged velocity:

$$\bar{u}(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} u(x,z,t)\, dz$$



An important player:

the Froude number

$$F = \frac{\bar{u}(x,t)}{\sqrt{h(x,t)\, g \cos\zeta}}$$

measures the relative importance of the

inertial forces vs. the gravitational force

Saint-Venant equations for granular chute flow

Mass conservation: $$\partial_t h + \partial_x \left(h\bar{u} \right) = 0$$

Momentum balance:

$$\partial_t \left(h\bar{u} \right) + \partial_x \left(h\bar{u}^2 \right) =$$

$$gh\sin\zeta - \mu(h,\bar{u})gh\cos\zeta - \tfrac{1}{2}\partial_x \left(gh^2 \cos\zeta \right) + \nu(\zeta)\,\partial_x \left(h^{3/2}\partial_x \bar{u} \right)$$

gravity component in x-direction

friction with the chute

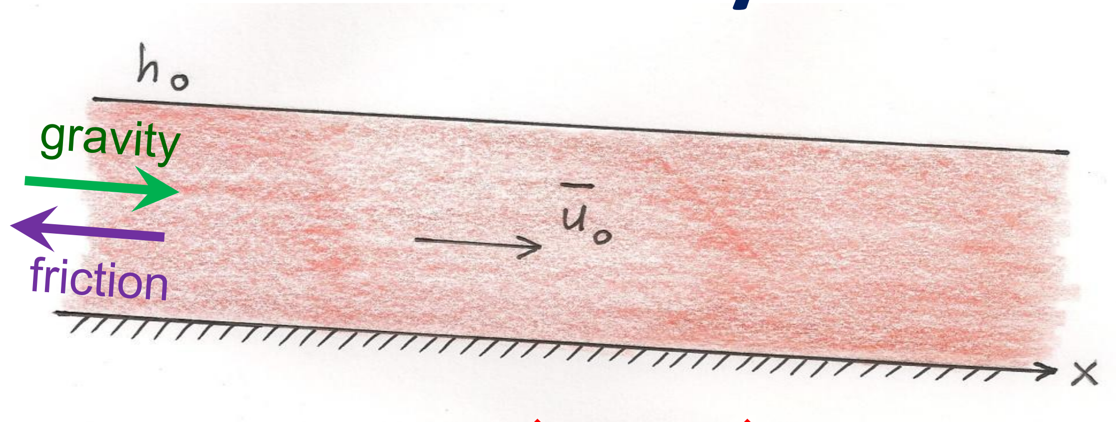
Pouliquen & Forterre, JFM **453** (2002)

gradient of depth-averaged pressure

viscous-like term

Gray & Edwards, JFM **755** (2014)

Basic solution: steady uniform flow



Mass conservation: $\cancel{\partial_t} h + \cancel{\partial_x}\left(h\bar{u}\right) = 0$ **trivially satisfied**

Momentum balance:

$$\cancel{\partial_t}\left(h\bar{u}\right) + \cancel{\partial_x}\left(h\bar{u}^2\right) =$$

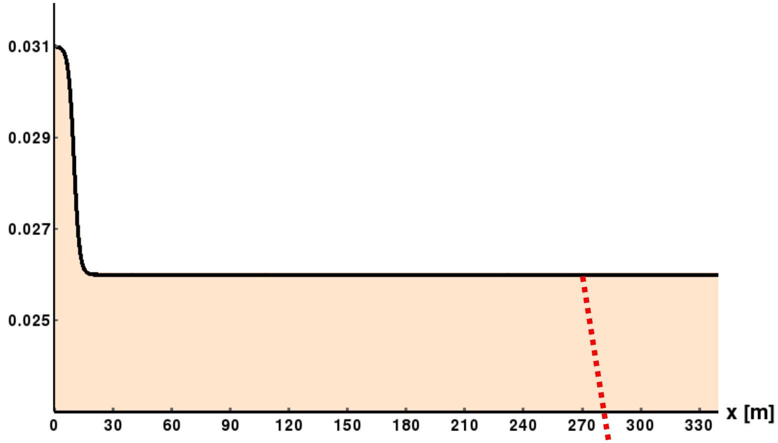
$$gh\sin\zeta - \mu(h,\bar{u})gh\cos\zeta - \tfrac{1}{2}\cancel{\partial_x}\left(gh^2\cos\zeta\right) + \nu(\zeta)\cancel{\partial_x}\left(h^{3/2}\partial_x\bar{u}\right)$$

gravity vs. friction

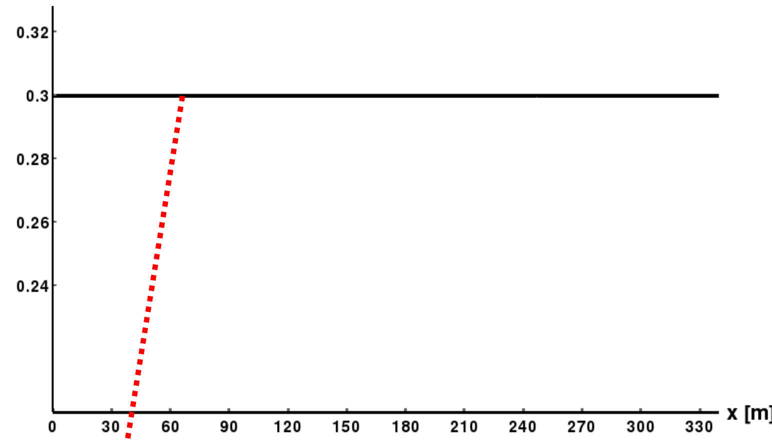
balance of two forces

Stable for $\beta < F < 2/3$

For the same range of F also the combination of uniform flows is stable



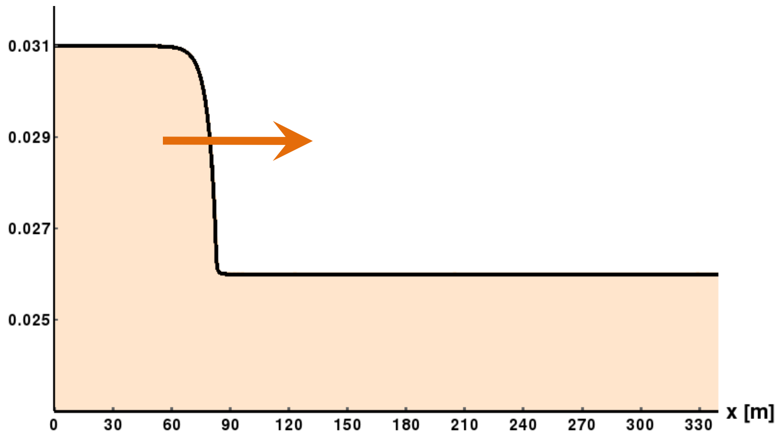
$\boldsymbol{h(x,t)}$ [m]



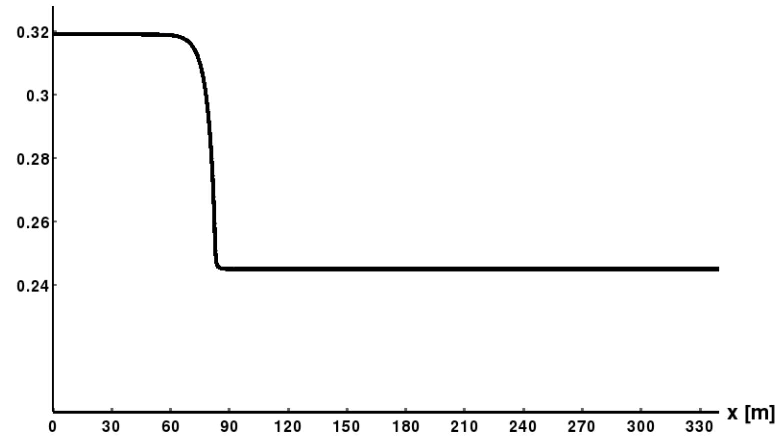
$\boldsymbol{\bar{u}(x,t)}$ [m/s]

$t = 0$ s

To illustrate this, we perform a numerical experiment with these initial conditions $h(x,0)$ and $\bar{u}(x,0)$



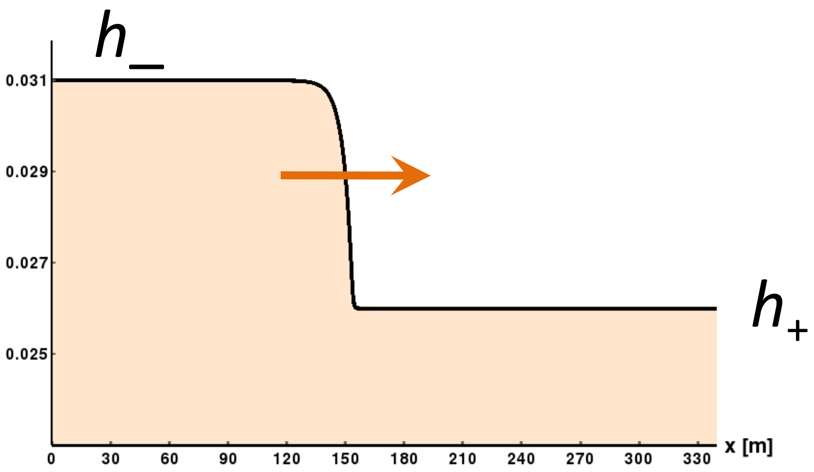
$h(x,t)$ [m]



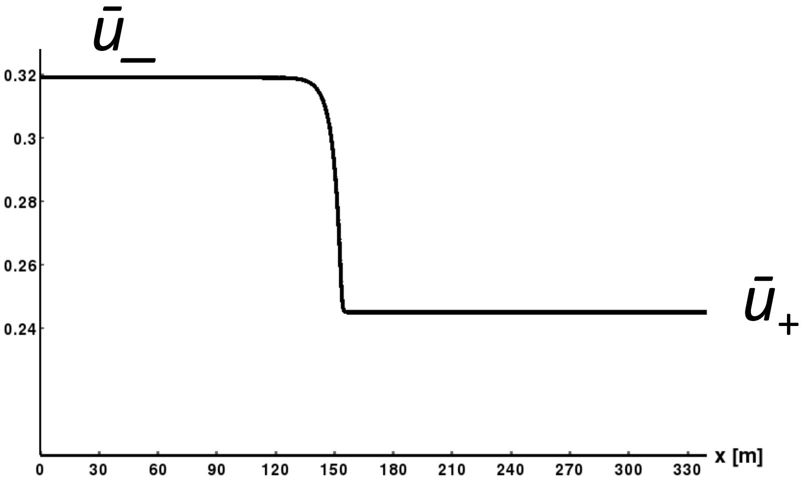
$\bar{u}(x,t)$ [m/s]

t = 100 s

The flow spontaneously organizes itself
in the form of a traveling **monoclinal wave**.

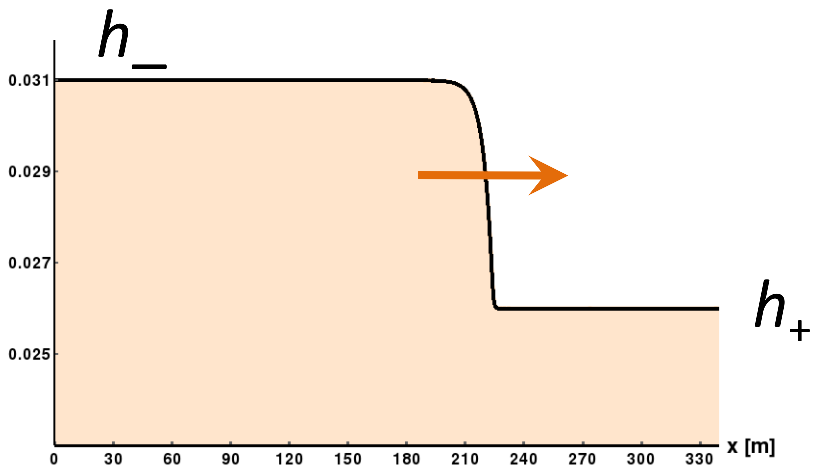


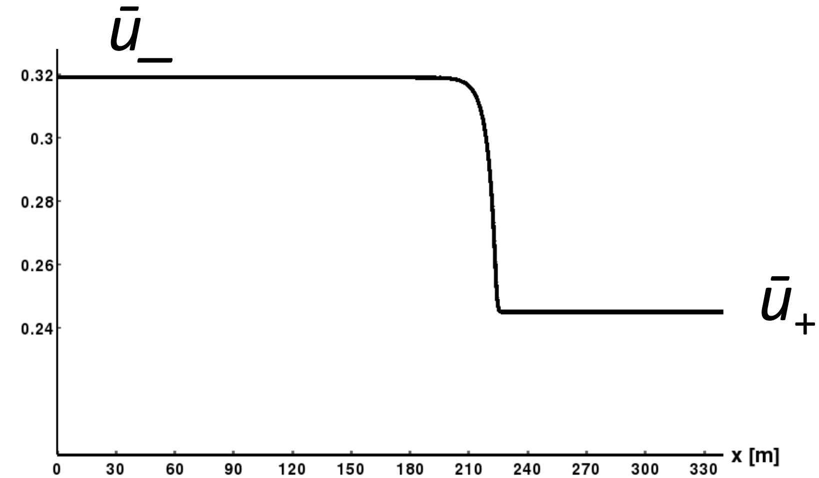
$\boldsymbol{h(x,t)}$ [m]



$\boldsymbol{\bar{u}(x,t)}$ [m/s]

t = 200 s





***h*(*x*,*t*)** [m]

***ū*(*x*,*t*)** [m/s]

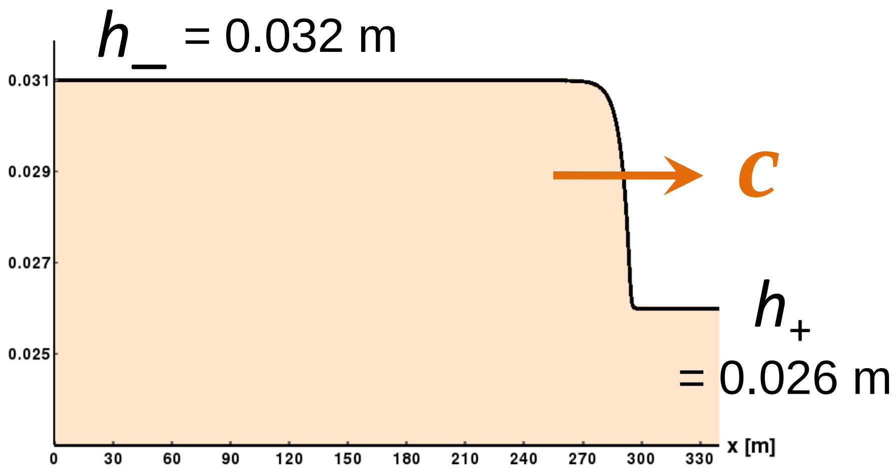
t = 300 s

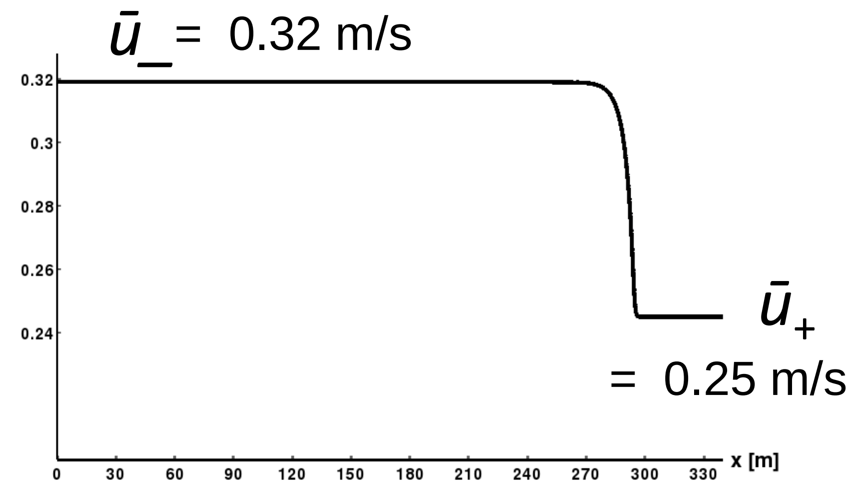
new granular waveform

J. Fluid Mech. (2018), *vol.* 843, *pp.* 810–846.
doi:10.1017/jfm.2018.149

The granular monoclinal wave

Dimitrios Razis[1], Giorgos Kanellopoulos[1,2] and Ko van der Weele[1,†]





$\boldsymbol{h(x,t)}$ [m]

$\boldsymbol{\bar{u}(x,t)}$ [m/s]

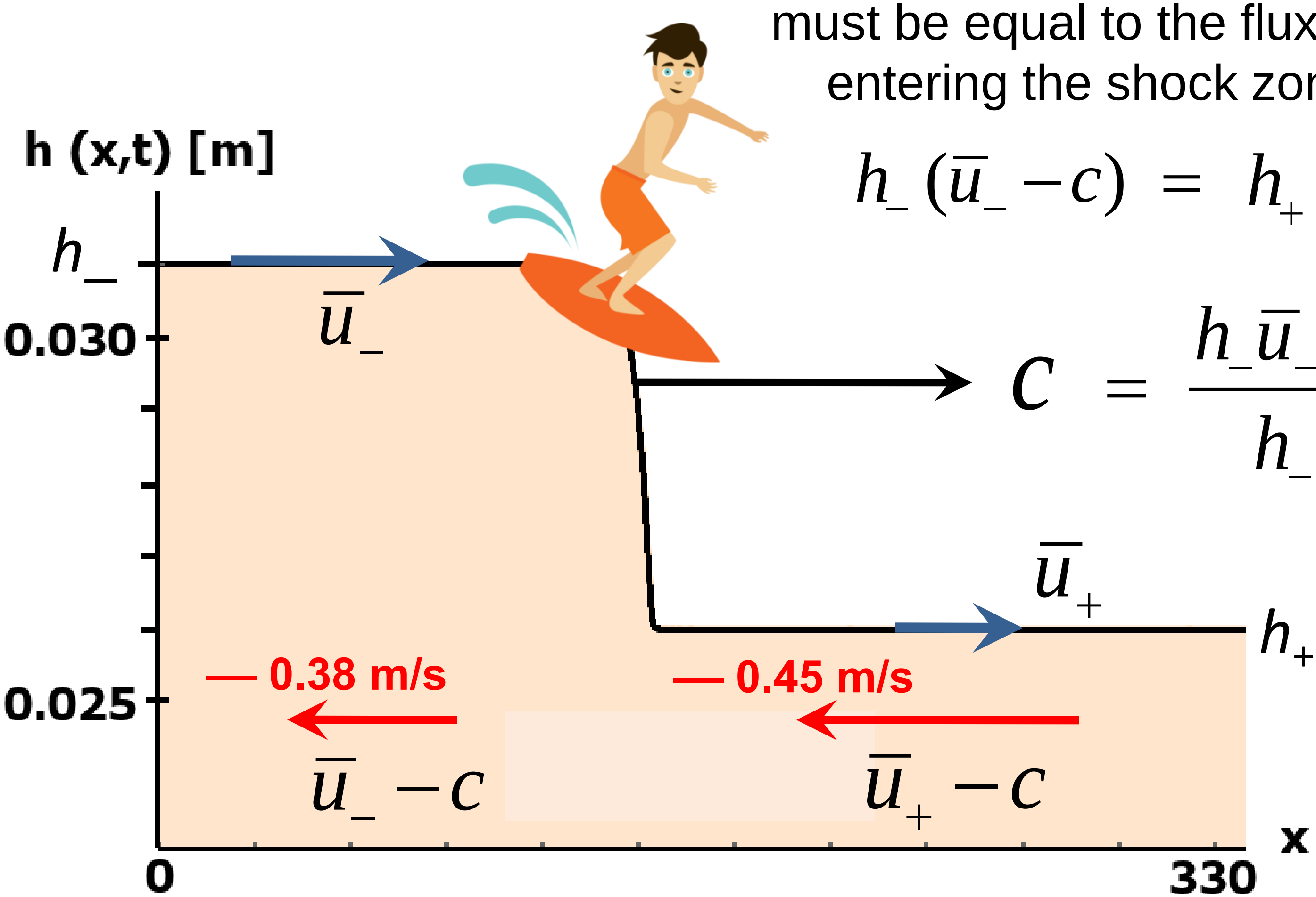
t = 400 s

wave speed: $c = 0.70$ m/s

Wave speed:

For an observer in the co-moving frame, the flux of material leaving the shock zone must be equal to the flux entering the shock zone:

$$h_-(\bar{u}_- - c) = h_+(\bar{u}_+ - c)$$



$$c = \frac{h_-\bar{u}_- - h_+\bar{u}_+}{h_- - h_+}$$

$$= 0.70 \text{ m/s}$$

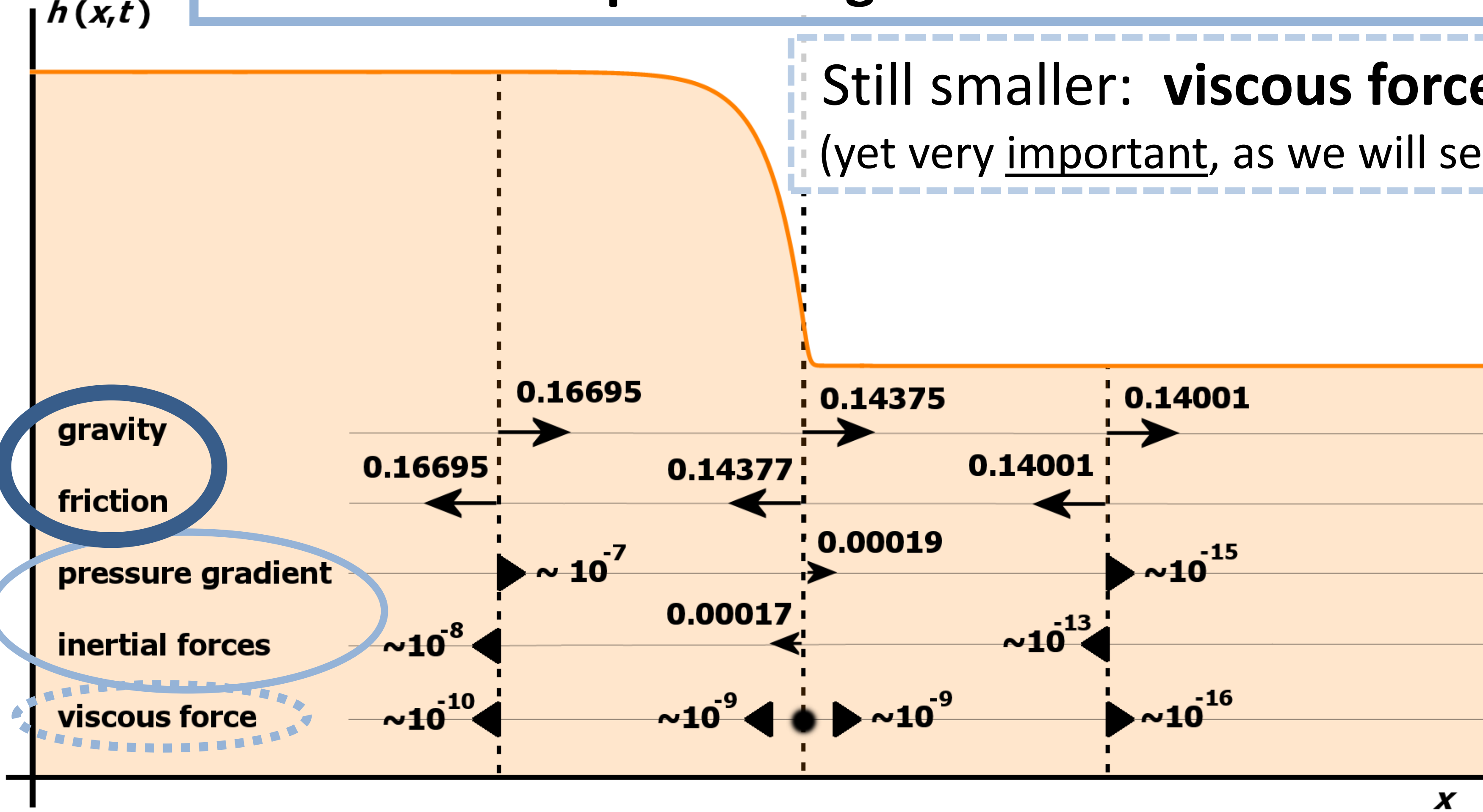
Balance of forces:

Main contributors: **gravity** & **friction**

Much smaller: **pressure gradient** & **inertial forces**

Still smaller: **viscous force**

(yet very <u>important</u>, as we will see)



Traveling wave analysis

We introduce the traveling-wave variable

$$\xi = x - ct$$

and are interested in solutions of the form

$$h(x,t) = h(x-ct) = h(\xi)$$

$$\bar{u}(x,t) = \bar{u}(x-ct) = \bar{u}(\xi)$$

The mass conservation then becomes:

$$-c\frac{dh}{d\xi} + \frac{d}{d\xi}\left(h\bar{u}\right) = 0$$

$$\boxed{\text{or: } -c\,h' + \left(h\bar{u}\right)' = 0}$$

This can be integrated immediately:

$$-c\,h' + \left(h\bar{u}\right)' = 0 \quad \Rightarrow \quad -ch + h\bar{u} = \boxed{-K}$$

integration constant

$h(\bar{u} - c)$ is the constant **flux** of material observed in the co-moving frame

With this ($\bar{u} = c - K\,h^{-1}$ and hence $\bar{u}\,' = K\,h^{-2}h'$, etc.) we can eliminate $\bar{u}$ and its derivatives from the <u>momentum balance</u>, which then takes the form

$$\frac{\nu K}{h^{3/2}} h'' - \frac{\nu K}{2\,h^{5/2}} (h')^2 + \left(\frac{K^2}{h^3} - g\cos\zeta \right) h'$$

$$+ g\sin\zeta - \mu(h) g\cos\zeta = 0$$

! Note that in the absence of viscosity ($\nu = 0$) the ODE would only be of 1st order.

This 2nd order ODE for $h(\zeta)$,
with the proper boundary conditions,
governs ***all*** traveling waveforms on the chute:

$$\boxed{\frac{\nu K}{h^{3/2}} h'' - \frac{\nu K}{2h^{5/2}} (h')^2 + \left(\frac{K^2}{h^3} - g \cos \zeta \right) h' + g \sin \zeta - \mu(h) g \cos \zeta = 0}$$

Dynamical Systems approach

The second-order ODE can be written as a system of 2 first-order equations:

$$\begin{cases} h' = s \\ s' = \ldots \end{cases}$$

s denotes the slope of $h(\xi)$

or non-dimensionally: with all length scales measured in units of the thickness $h\_$ of the incoming stream

$$\begin{cases} \dfrac{d\tilde{h}}{d\tilde{\xi}} = \tilde{s} \\[2ex] \dfrac{d\tilde{s}}{d\tilde{\xi}} = \dfrac{\tilde{s}^2}{2\tilde{h}} - \dfrac{9\tilde{h}^{3/2}}{2\tan\zeta(\tilde{c}-1)}\left[\left(\dfrac{F_{in}^2(\tilde{c}-1)^2}{\tilde{h}^3} - 1\right)\tilde{s} + \tan\zeta - \mu(\tilde{h})\right] \end{cases}$$

Instead of $\tilde{c}$ one may also choose $\tilde{h}_+$,

since $\tilde{c} = \tilde{c}(\tilde{h}_+) = \dfrac{1-\tilde{h}_+^{5/2}}{1-\tilde{h}_+}$

Three dimensionless parameters:

$\boldsymbol{F_{in}}$, $\tilde{C}$ and ζ

$$\begin{cases} \dfrac{d\tilde{h}}{d\tilde{\xi}} = \tilde{s} \boxed{= f(\cancel{\tilde{h}}, \tilde{s})} \\[2ex] \dfrac{d\tilde{s}}{d\tilde{\xi}} = \dfrac{\tilde{s}^2}{2\tilde{h}} - \dfrac{9\tilde{h}^{3/2}}{2\tan\zeta(\tilde{c}-1)}\left[\left(\dfrac{F_{in}^2(\tilde{c}-1)^2}{\tilde{h}^3} - 1\right)\tilde{s} + \tan\zeta - \mu(\tilde{h})\right] \end{cases}$$

$$\boxed{= g(\tilde{h}, \tilde{s})}$$

Fixed points:

fixed points correspond to **flat regions** of the flow!

$$\begin{cases} \dfrac{d\tilde{h}}{d\tilde{\xi}} = 0 \quad \longrightarrow \quad f(\tilde{s}) = \tilde{s} = 0 \\ \dfrac{d\tilde{s}}{d\tilde{\xi}} = 0 \quad \longrightarrow \quad g(\tilde{h},\tilde{s}) = g(\tilde{h},0) = 0 \end{cases}$$

→ **two** fixed points: $\boxed{(\tilde{h}_+,0)}$ and $\boxed{(\tilde{h}_-,0)=(1,0)}$

… and their stability:

determined by the eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial f(\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial f(\tilde{s})}{\partial \tilde{s}} \\ \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{s}} \end{pmatrix}_{(\tilde{h}_\pm,0)} = \begin{pmatrix} 0 & 1 \\ \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{s}} \end{pmatrix}_{(\tilde{h}_\pm,0)}$$

Eigenvalues for the two fixed points:

$\boxed{(\tilde{h}_+, 0)}$

$\boxed{(\tilde{h}_-, 0) = (1, 0)}$

$$\lambda_{a,b}^{(\tilde{h}_+, 0)}(\tilde{h}_+, F, \zeta)$$

$$\lambda_{a,b}^{(1,0)}(\tilde{h}_+, F, \zeta)$$

$\downarrow$

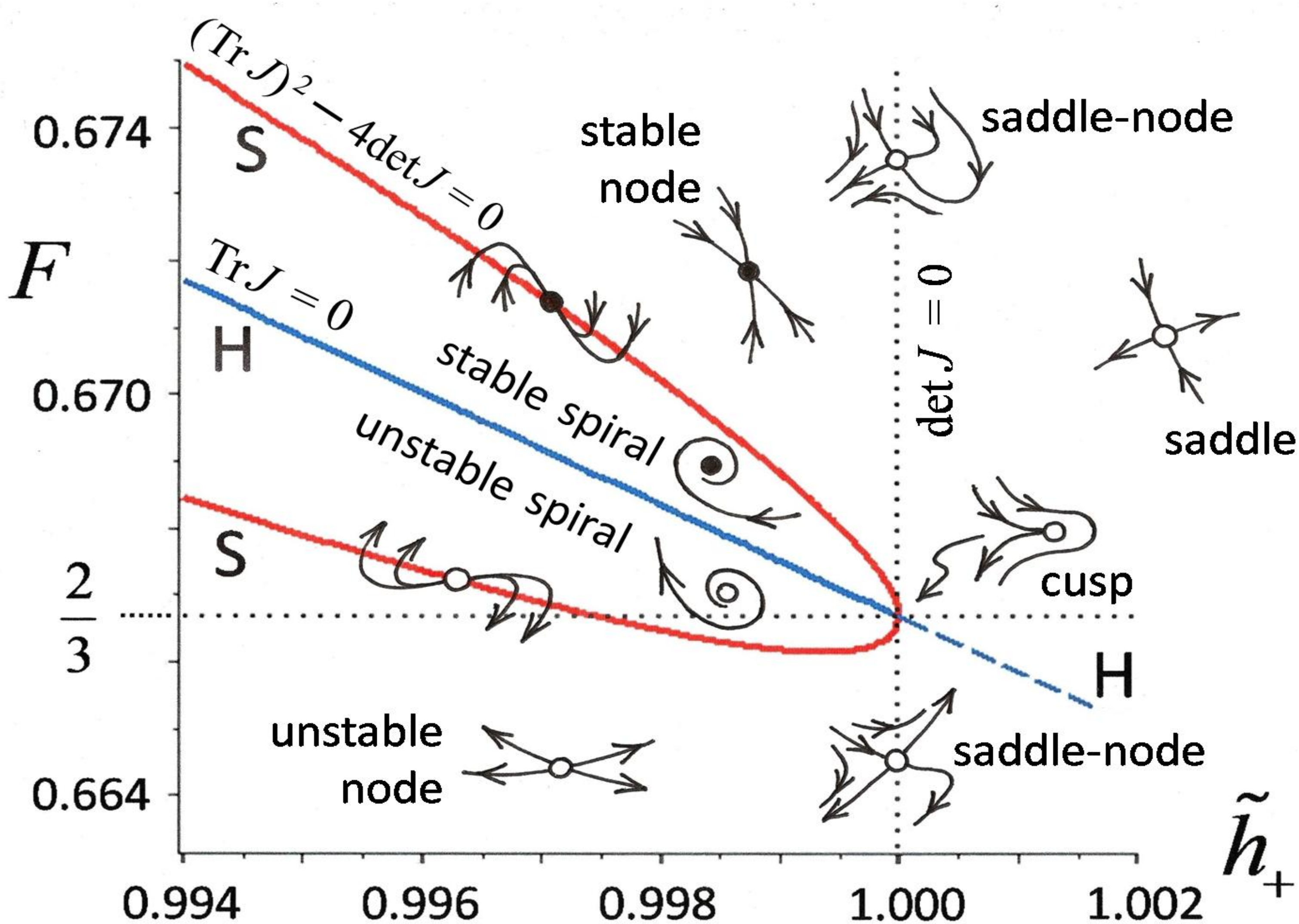
Real and of-opposite-sign for all relevant values of the system parameters.

→ So $(\tilde{h}_+, 0)$ is a **saddle**.

$\downarrow$

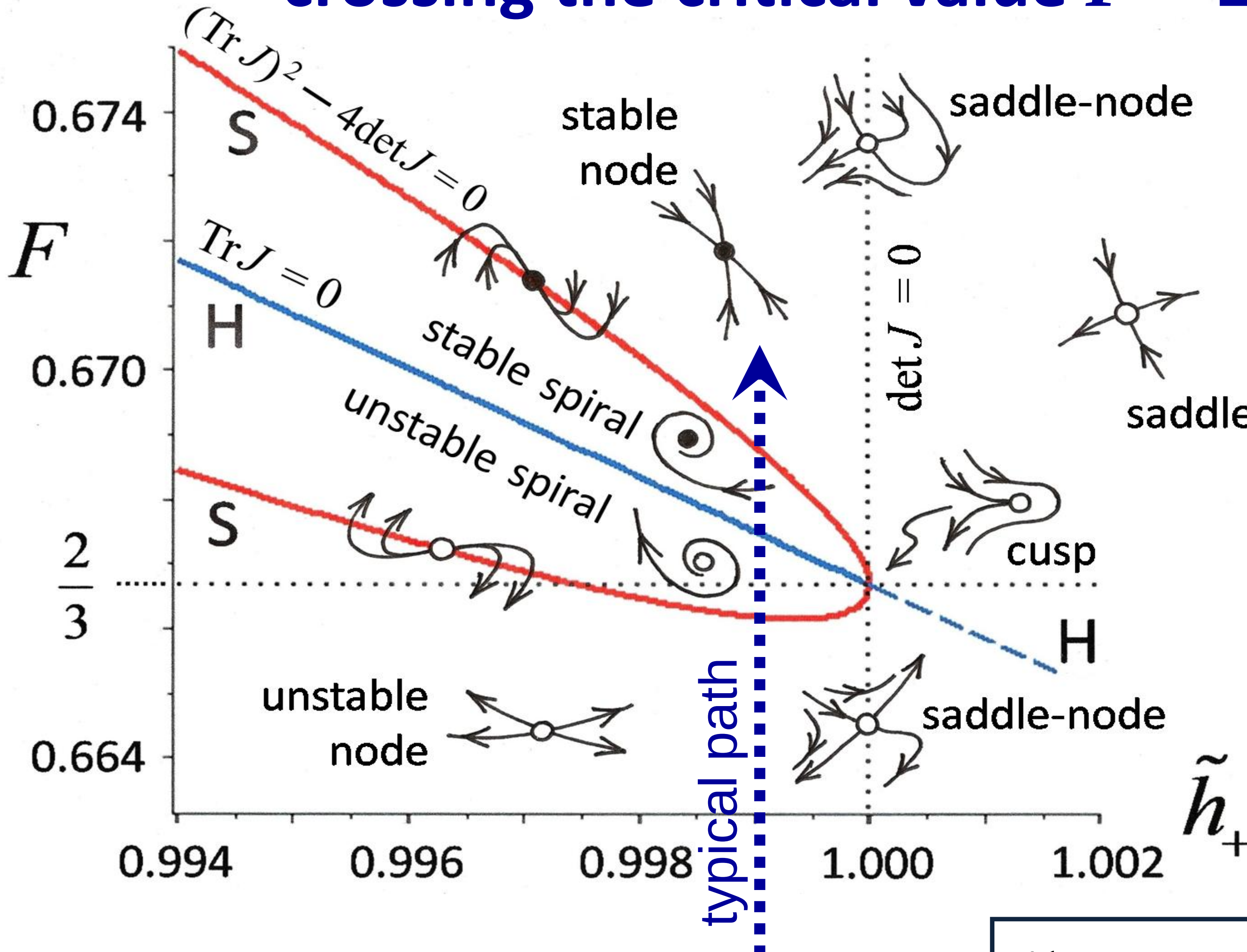
More versatile:
$(1, 0)$ can be ***any*** type of fixed point, depending on the system parameters.

Fixed point (1,0) depending on F and $\tilde{h}_+$:



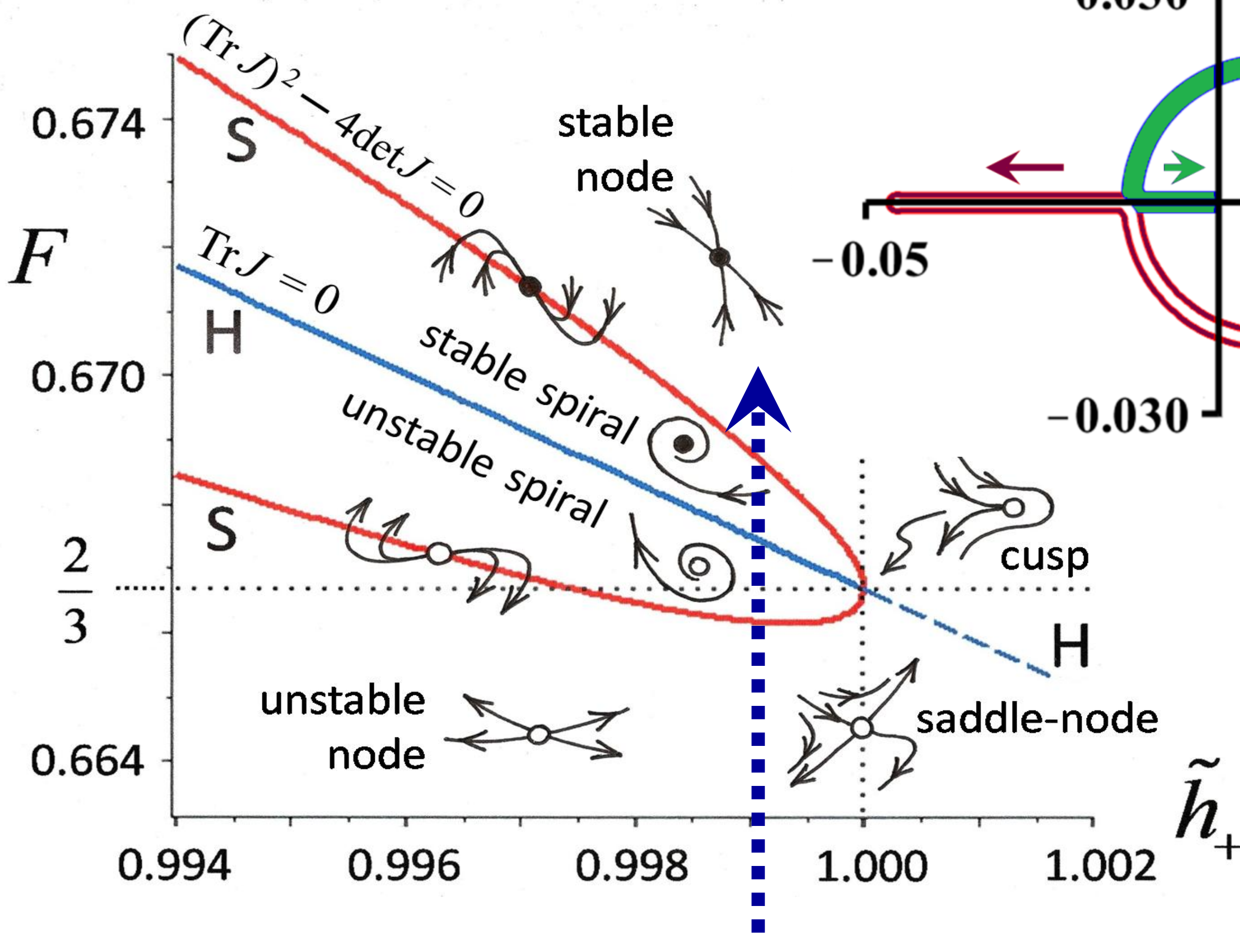
(ζ = 33.3 degrees)

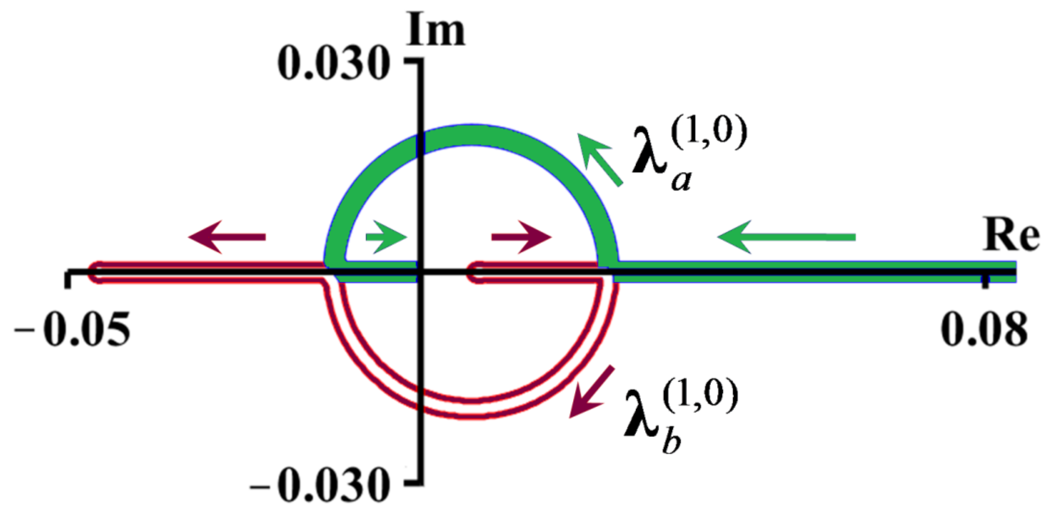
Typical path through the parameter diagram, crossing the critical value $F = 2/3$



(ζ = 33.3 degrees)

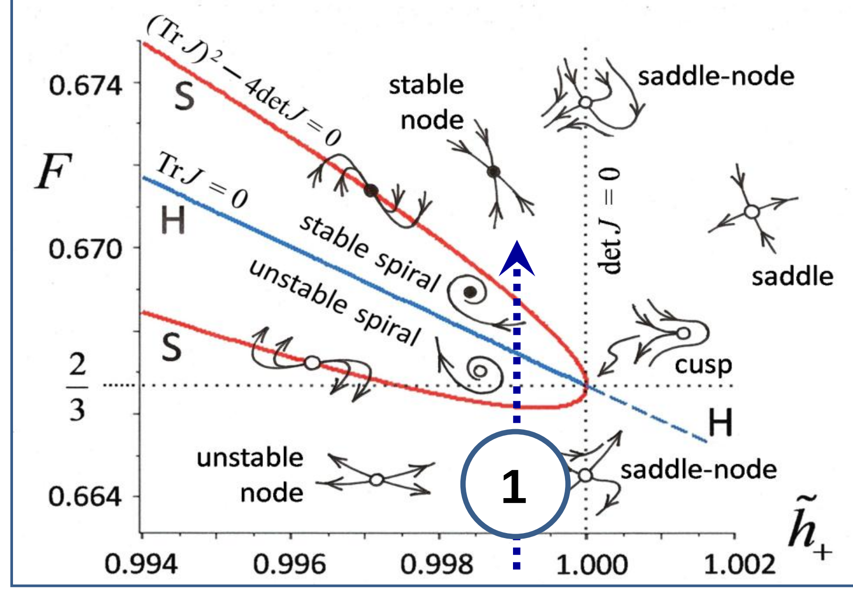
Eigenvalues of (1,0) along the path:





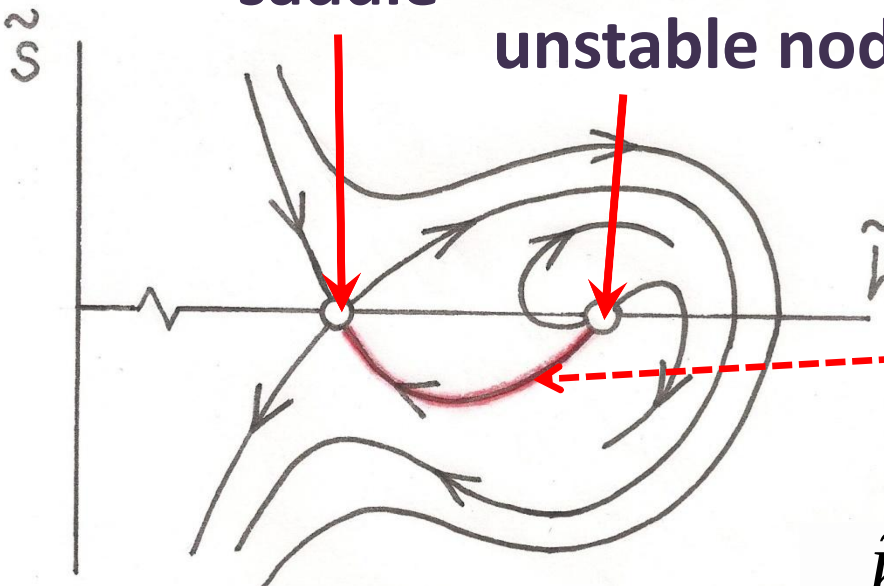
(ζ = 33.3 degrees)

Stage 1:



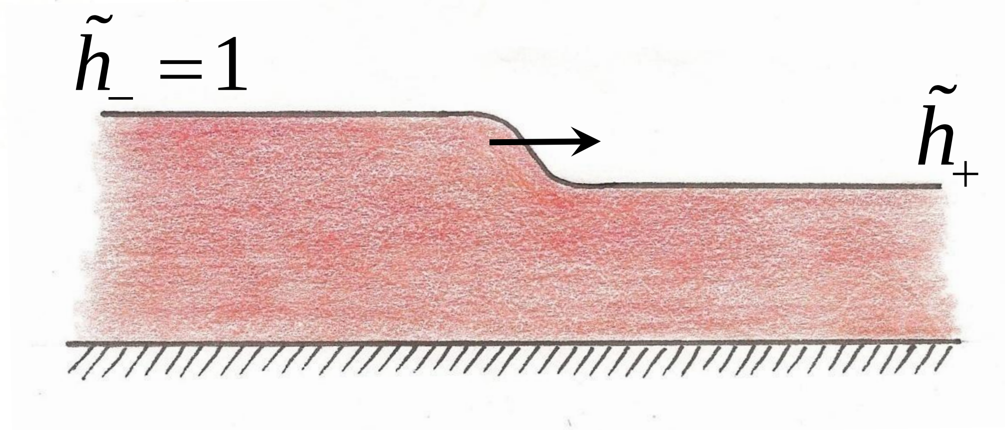
$(\tilde{h}_+,0)$ saddle

$(\tilde{h}_-,0)=(1,0)$ unstable node



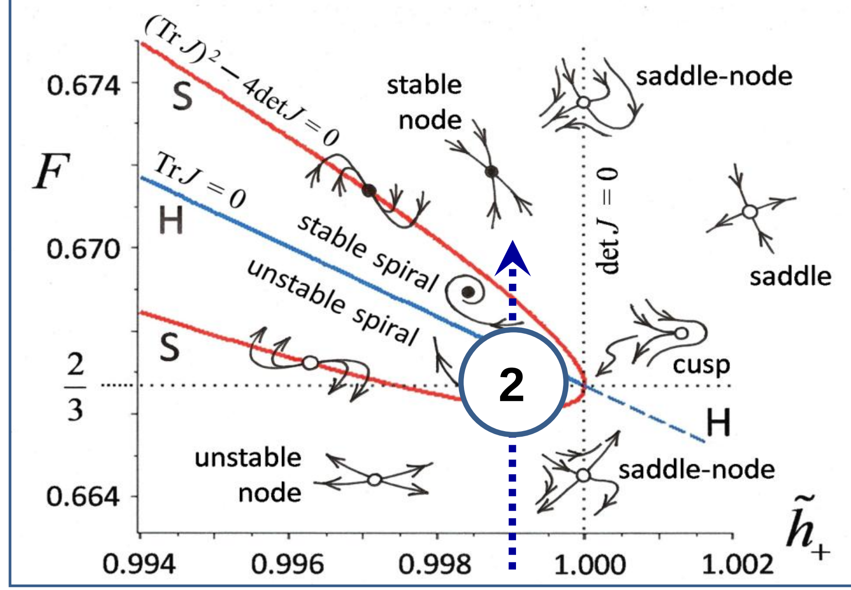
heteroclinic connection

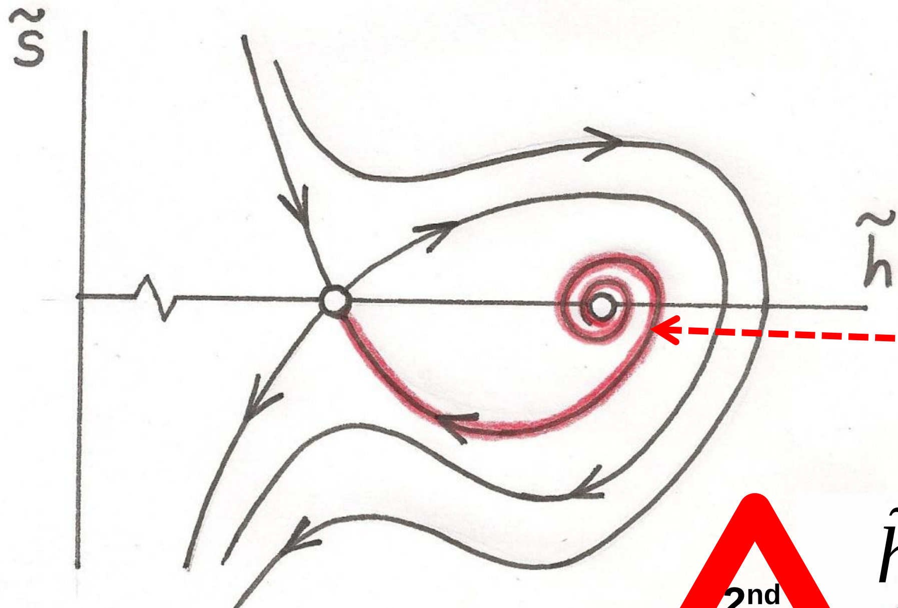
= monoclinal wave



Stage 2:

(1,0) becomes an unstable spiral

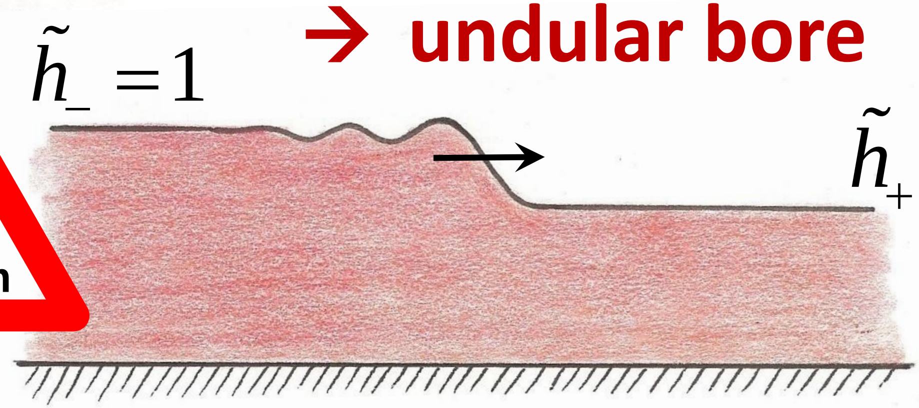




The heteroclinic connection now <u>spirals</u> around $(1,0)$

→ undular bore

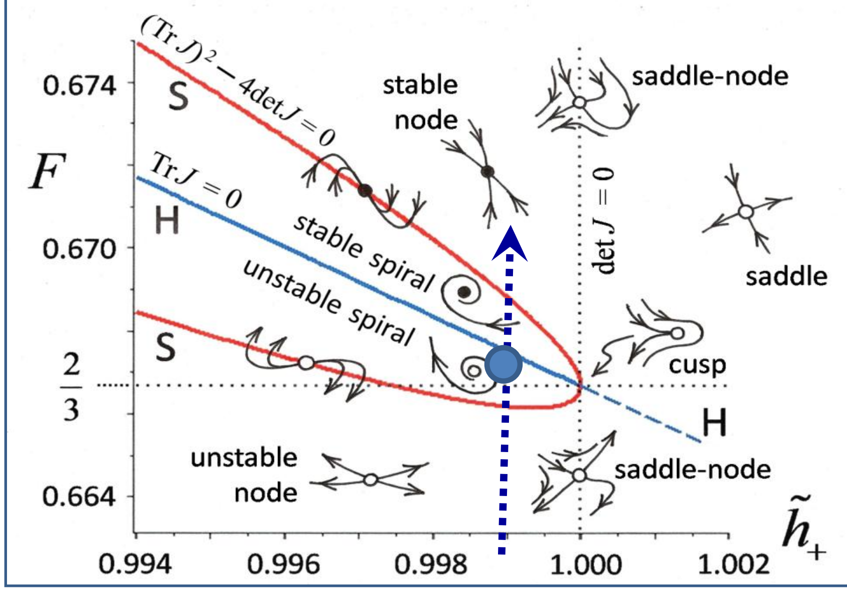
2nd new granular waveform

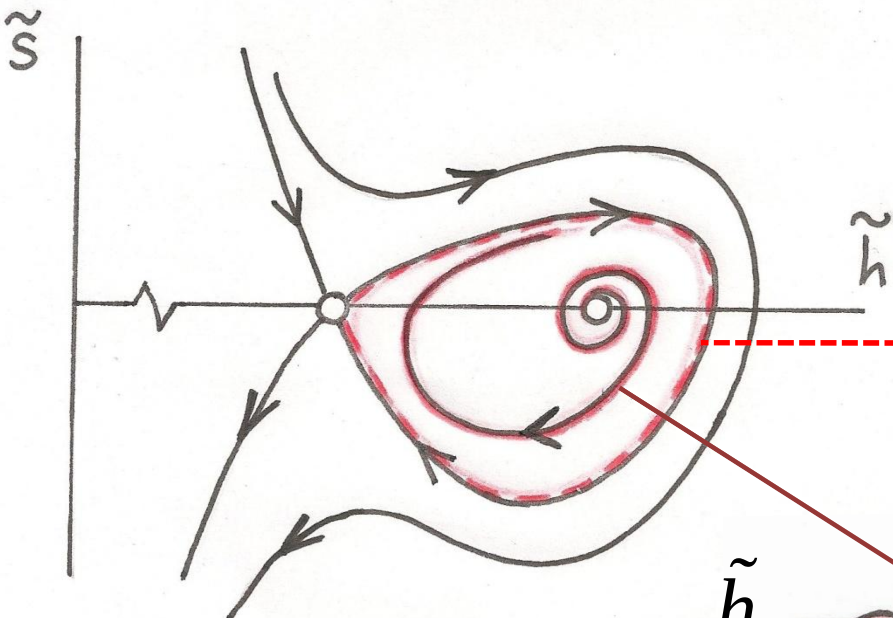


Intermediate event:

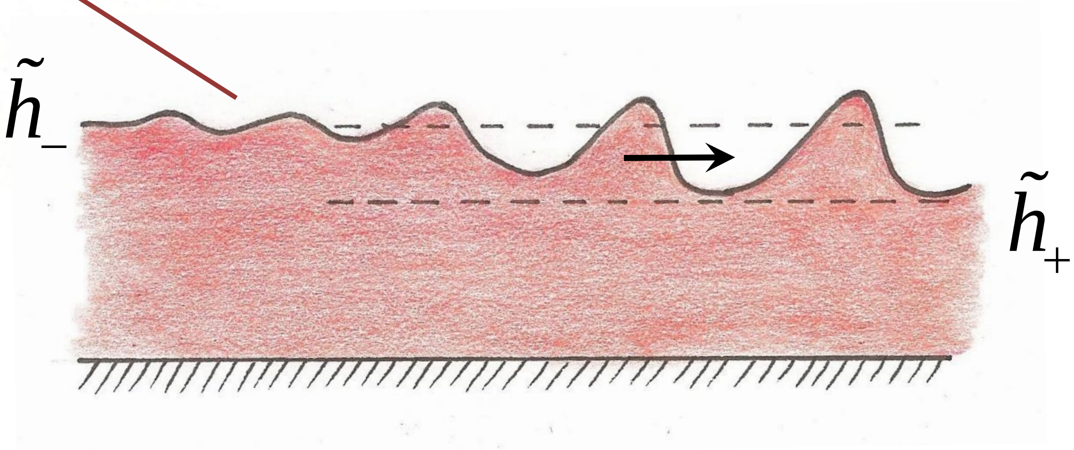
a saddle-loop

bifurcation!



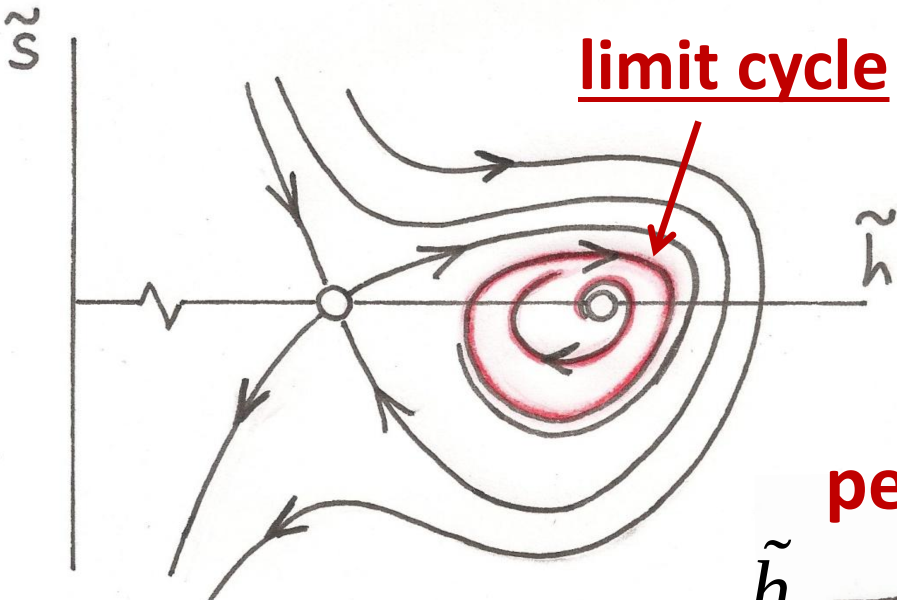


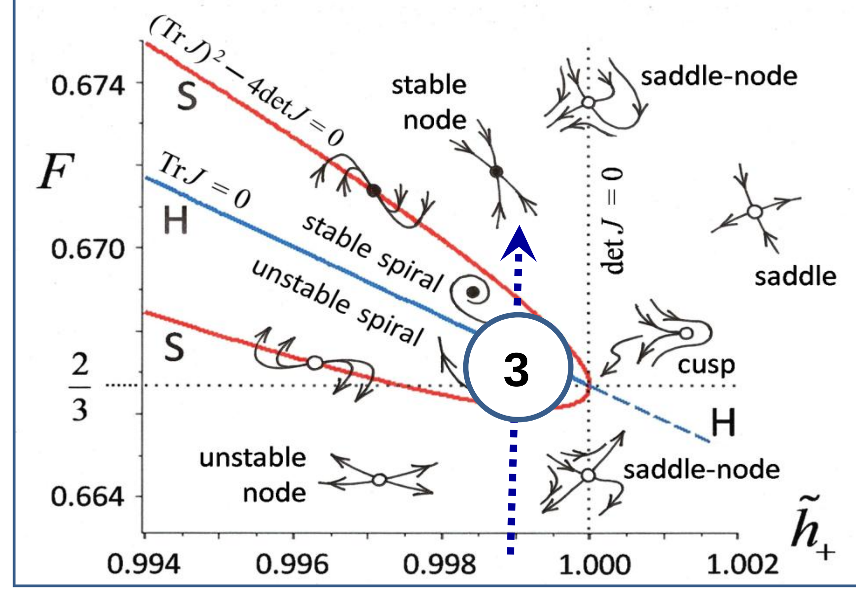
The saddle-loop corresponds to a solitary roll wave



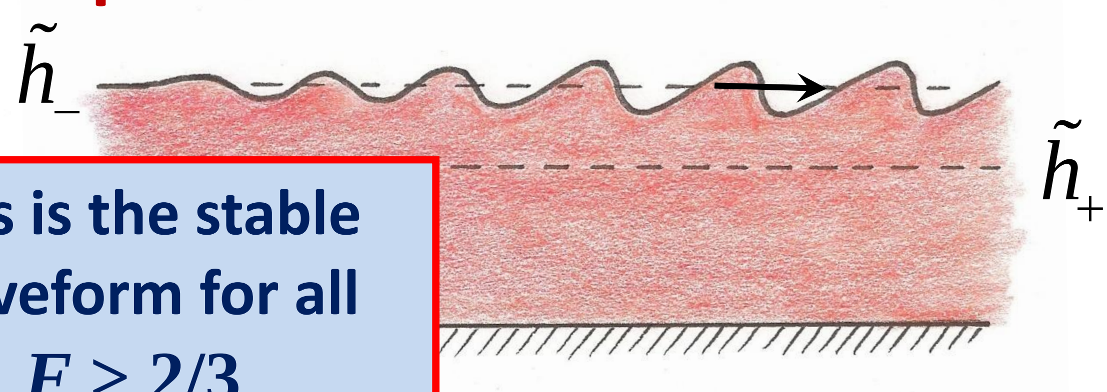
Stage 3:

The saddle-loop has evolved into a stable limit cycle

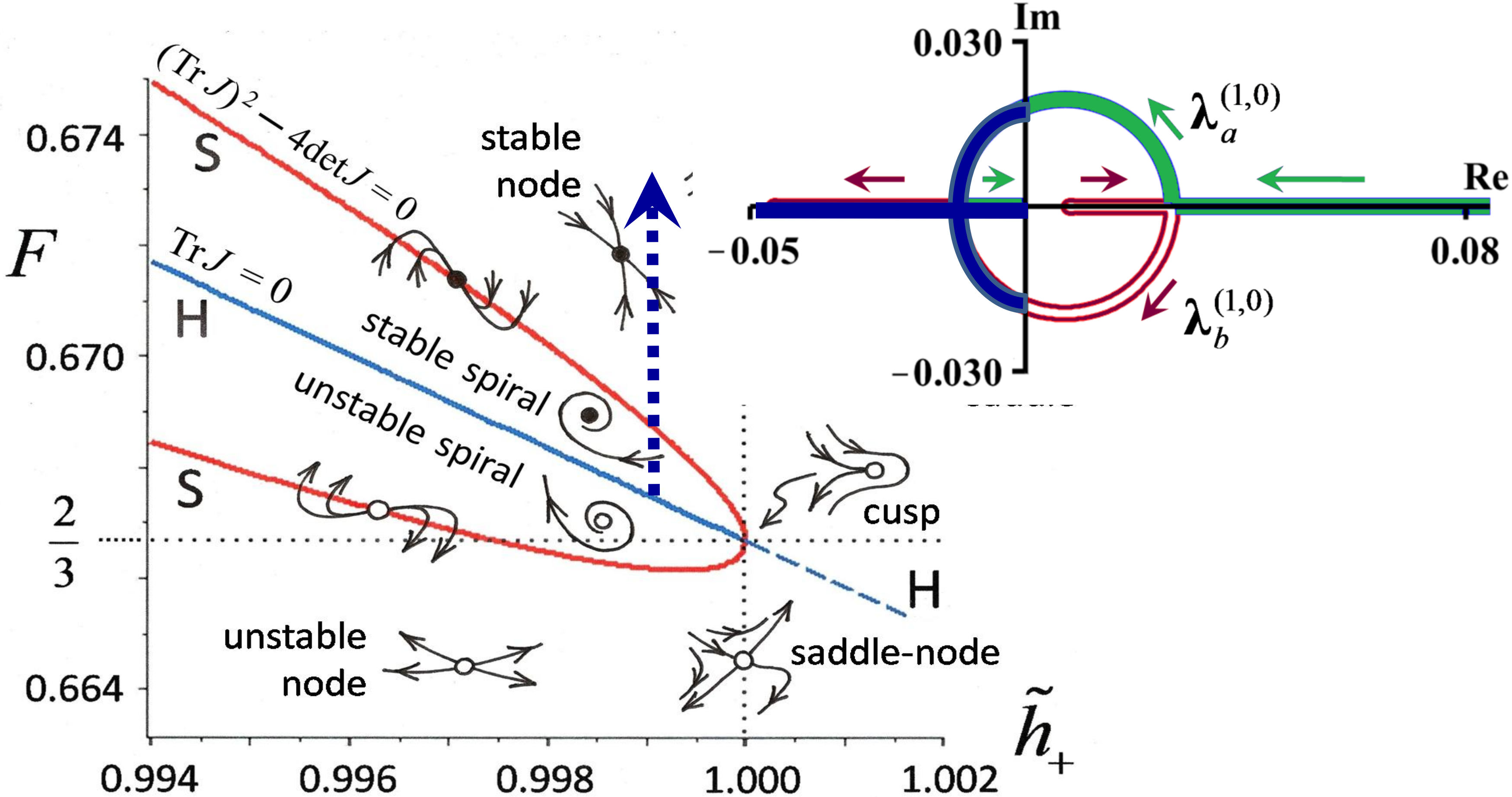




…, corresponding to a periodic train of roll waves:

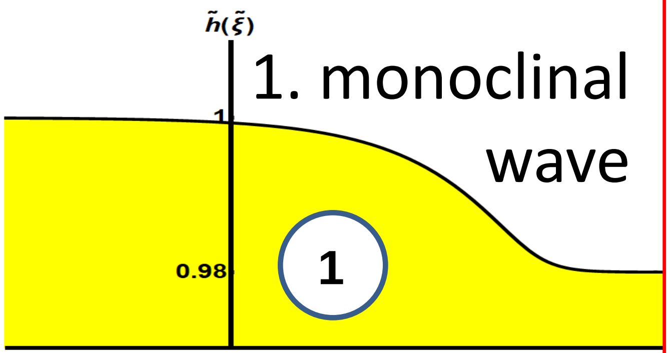


This is the stable waveform for all $F > 2/3$



The next stages are mathematically interesting (involving a Hopf bifurcation etc.) but yield only <u>unstable</u> waveforms.

So we arrive at the following transition scenario:



1. monoclinal wave

J. Fluid Mech. (2019), vol. 869, pp. 143–181. © Cambridge University Press 2019 143
doi:10.1017/jfm.2019.168

A dynamical systems view of granular flow: from monoclinal flood waves to roll waves

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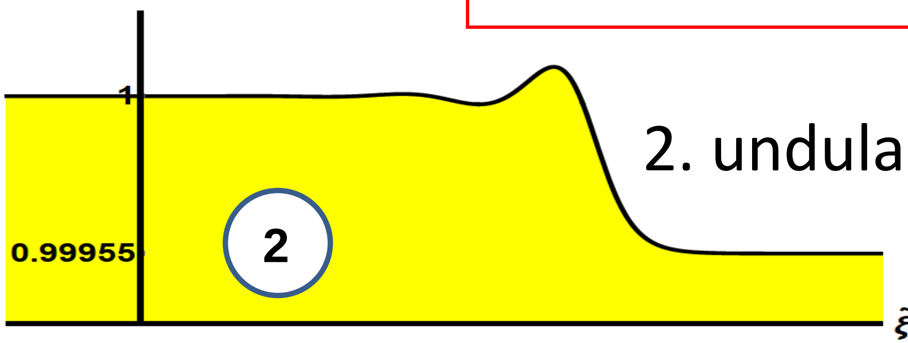
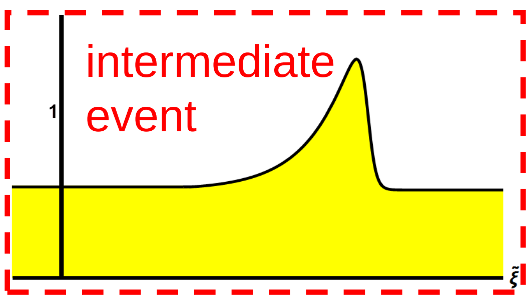
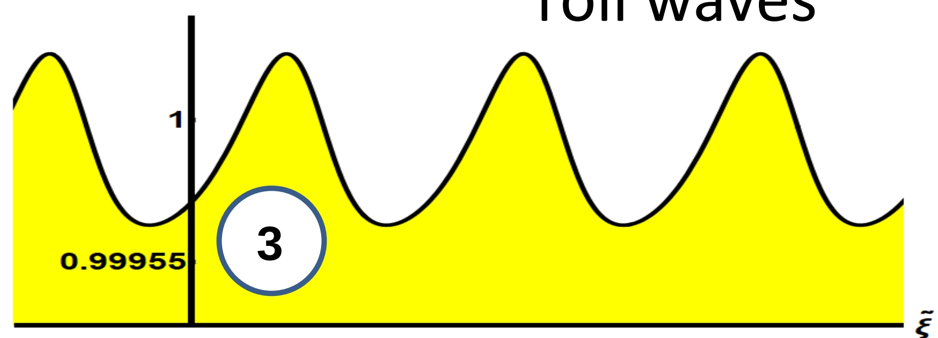


2. undular bore



3. periodic roll waves



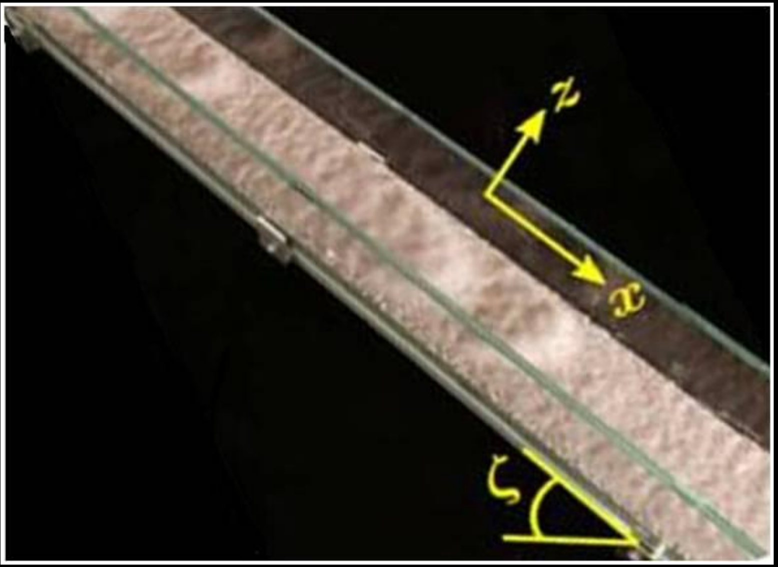
Conclusion

A. The **Dynamical Systems** approach is a powerful tool for analyzing the waves that may be encountered in granular chute flow.

B. It has revealed a whole **spectrum of waveforms** that were hitherto unknown in granular flow:
● monoclinal flood wave ● undular bore
● solitary roll wave, ● and various unstable ones.

C. For growing *F*, we predict the **transition**
monoclinal wave → undular bore → roll waves

D. The challenge is now to verify this **in experiment**.



The End