Hamiltonian Dynamics for Stellarator Design

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1. Stellarators

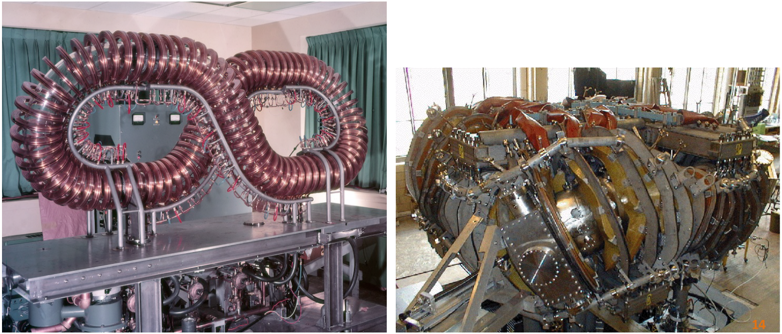


Figure: Some stellarators: Spitzer, HSX

Magnetic confinement devices for plasma (ionised gas), in which the magnetic field rotates around a closed field line because of non-trivial 3D geometry.

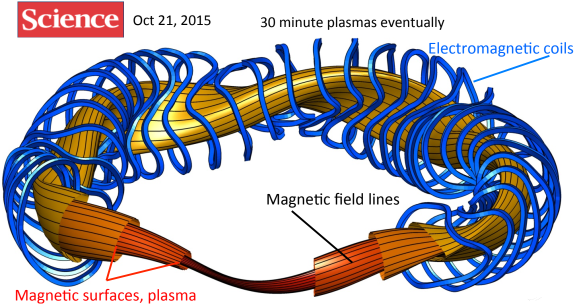


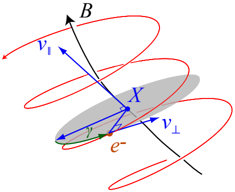
Figure: Schematic of Wendelstein 7X

In contrast to tokamaks, which are near axisymmetric but require a strong toroidal current to make the field rotate around a central fieldline; this current causes instabilities and needs driving.

Catch with stellarators is that they need more careful design to confine the plasma, but this makes interesting mathematics.

Magnetic confinement of charged particles

To leading order in $mv^2/e|B|$, charged particles in a magnetic field B describe helices around a fieldline.



Write position q and velocity v in terms of a guiding centre (GC) X, gyroradius vector $\rho \perp b = B/|B|$, and parallel velocity $v_\parallel$:

$$v = \frac{e}{m} B(X) \times \rho + v_\parallel b(X), \quad q = X + \rho.$$

ρ rotates at gyro-frequency $\Omega_g = e|B|/m$ and magnetic moment $\mu = mv_\perp^2/2|B|$ is an adiabatic invariant.

Zeroth-order GC motion (ZGCM): X moves along a fieldline at velocity $v_{\|}$, conserving $H = \frac{1}{2}v_{\|}^2 + \mu|B(X)|$, so circulating in one direction if energy H is large enough, else bouncing at places where $|B| = H/\mu$.

At first order, X drifts across non-uniform B, but can confine it by various designs.

In particular, if GC motion has another integral K, independent of H, with bounded joint level sets, then GCs are confined (ignoring their interactions).

This is equivalent to having a continuous symmetry u and a topological condition to make K single-valued. u is called a *quasi-symmetry* (q-s).

Axisymmetry would do, but to make the joint level sets bounded requires toroidal current.

Hamiltonian formulation

State space $\{(X, v_\parallel) : X \in \mathbb{R}^3, v_\parallel \in \mathbb{R}\}$. Inner product $\cdot$ and compatible $\times$ and volume form Ω on $\mathbb{R}^3$.

Hamiltonian $H = \frac{1}{2} m v_\parallel^2 + \mu |B(X)|$.

Symplectic form $\omega = e\beta + d(m v_\parallel b^\flat)$,
where $\beta = i_B \Omega$, i.e. $\beta(\xi, \eta) = \Omega(B, \xi, \eta)$, and $b^\flat(\xi) = b \cdot \xi$.

Defines first-order GC motion (FGCM) Hamiltonian vector field V by $i_V \omega = dH$.

Produces usual Littlejohn drift equations

$$\dot{X} = (v_\parallel \tilde{B} + \frac{\mu}{e} b \times \nabla |B|)/\tilde{B}_\parallel, \quad \dot{v}_\parallel = -\frac{\mu}{m} \frac{\tilde{B}}{\tilde{B}_\parallel} \cdot \nabla |B|,$$

where $\tilde{B} = B + \frac{m}{e} v_\parallel c$, with $c = \mathrm{curl} b$.

Theorem [Burby, Kallinikos, M, arXiv:1912.06468]

Theorem: $(u, 0)$ is a quasi-symmetry of FGCM for all μ iff

$$L_u|B| = 0, L_u\beta = 0, L_u B^\flat = 0,$$

where L_u is the Lie derivative along u. In vector calculus, $u \cdot \nabla|B| = 0, \mathrm{curl}(B \times u) = 0, u \times J = \nabla(u \cdot B)$, with $J = \mathrm{curl}B$.

Furthermore,

$\mathrm{curl}(B \times u) = 0 \Rightarrow B \times u = \nabla\psi$ for some local function ψ;

if ψ is global then FGCM has integral $K = -e\psi - mv_{||}u \cdot b$;

conservation of $H\&K$ implies the value of ψ at the GC cannot change much for moderate energies, hence confinement.

Can q-s be achieved (apart from axisymmetry)?:
Are there Kovalevskaya examples?
Or does every q-s have to be a Killing field, $L_u g = 0$?

Perhaps approximate q-s suffices, especially as FGCM is only first order.

Or weaker designs, e.g., omnigenity: $B \cdot \nabla\psi = 0$, $\langle\dot{\psi}\rangle = 0$ for FGCM averaged along ZGCM.

Or "property X": $L = \int_\gamma eA^\flat + mv_\parallel b^\flat$ along closed orbits γ of ZGCM is constant on the set of internal tangencies: $B \cdot \nabla|B| = 0$, $D^2|B|(B, B) < 0$.

Or for non-interacting GCs, one invariant torus for each value of (μ, H) suffices.

2. Region occupied by invariant tori

KAM theory gives sufficient conditions for existence of invariant tori and bounds on where they are, but it is hard.

Much easier is Converse KAM theory, sufficient conditions for non-existence of invariant tori of given class through a given region.

e.g. I used this in 1983 to make a computer-assisted proof that the standard map has no rotational invariant circles for all $k \geq 63/64$.

I've adapted the method to establish regions through which pass no invariant tori transverse to a given foliation.

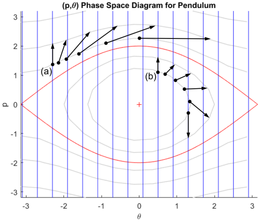
Plan to apply it to FGCM for some example magnetic fields.
But in the meantime, applied it to the planar circular restricted 3-body problem, with Tom Syndercombe.

Simple example: Pendulum

$$H(\theta, p) = \frac{1}{2}p^2 - \cos\theta.$$

$H > 1$: Rotational invariant tori, transverse to foliation $\mathcal{F}$, $\theta = \text{cst}$. Orbit of upward tangent to $\mathcal{F}$ cannot cross the tangent to the torus so cannot cross the downward tangent to $\mathcal{F}$.

$H < 1$: Librational invariant tori. Can tell they are not on invariant tori transverse to $\mathcal{F}$ because the orbit of upward tangent to $\mathcal{F}$ crosses the downward tangent after some time.



We will use a 2DoF extension of this idea: M, RegChDyn 23 (2018) 797; building on M&Percival CMP (1985), M Phys D (1989).

Planar Circular Restricted Three-Body Problem

Two masses $1-\mu$ and μ, in circular orbits round their centre of mass, separation 1, frequency 1.

Coordinates (x, y) in rotating frame, masses at $(-\mu, 0)$ and $(1-\mu, 0)$.

Asteroid/planet of (relatively) negligible mass in their plane of rotation. Want to know where it can orbit "stably".

Let $\mathbf{p}, K, L$ be its momentum, energy and angular momentum (per unit mass) in the instantaneous inertial frame:

$$K = \frac{1}{2}|\mathbf{p}|^2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \quad L = xp_y - yp_x,$$

where r_1, r_2 are the distances to the primaries.

Then the motion in the rotating frame is given by Hamilton's equations for $H = K - L$. In particular, H is conserved and its value is denoted $-C/2$ with C called the Jacobi constant.

Invariant tori for $\mu = 0$

When $\mu = 0$, H is integrable, with integrals K and L.

The bounded regular level sets are 2-tori (points on Kepler ellipses with the same semi-major axis a, eccentricity e and orientation).

The motion on them is quasiperiodic: $\dot{m} = N^{-3}$, $\dot{g} = -1$, where m is 2π times fraction of area of ellipse swept out from pericentre, g is angle of pericentre, and $N = \sigma/\sqrt{-2K} = \sigma\sqrt{a}$ is principal quantum number, with $\sigma = \text{sign } L$.

The tori are $p_r^2 + L^2/r^2 - 2L - 2/r = -C$ for $-\frac{1}{2}L^{-2} < K < 0$.

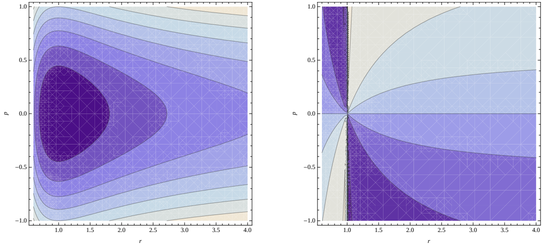


Figure: Invariant tori for $L = 1$ and Transverse foliation by $g =$cst.

Transverse foliation

The tori are transverse to the foliation $\mathcal{F}$: $g = \text{cst}$, $\theta = \text{cst}$, because (L, N, g, θ) form a local coordinate system.

The tori have frequency ratio $\rho = -\dot{g}/\dot{m} = N^3$.

By KAM theory, sufficiently irrational tori persist for some range $0 \leq \mu < \mu_c(\rho, L)$. They also remain transverse to $\mathcal{F}$ for μ small enough: (L, N) is a C^1 function of (g, θ). (Or for fixed μ, KAM theory applies for $r_{min} = N^2(1 - \sqrt{1 - L^2/N^2})$ large enough)

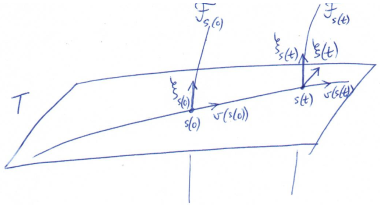
g is the direction of the Laplace vector $\mathbf{e} = \mathbf{p} \times L\hat{\mathbf{z}} - \hat{\mathbf{r}}$, i.e. solution of $e \cos g = p_y L - x/r, e \sin g = -p_x L - y/r$ with $e \geq 0$.

Restricting to $H = -C/2$, $\mathcal{F}$ becomes 1D and we choose a continuous field of tangent vectors ξ to it (except at $e = 0$ where $\mathcal{F}$ is singular), so that $dL\,\xi > 0$ (except on $L = 0$).

We choose a 1-form $\varepsilon(\mu)$ so that $\lambda = dL + \varepsilon$ satisfies $\lambda\,\xi > 0$ and $\lambda\,v = 0$, where v is the Hamiltonian vector field.

Non-existence condition

If $\exists$ invariant torus T through $s(0) \in H^{-1}(-C/2)$ transverse to $\mathcal{F}$, take $\xi(0) = \xi_{s(0)}$ and simultaneously evolve $s(t)$ using dynamics $\dot{s} = v(s)$ and $\xi(t)$ using the linearised dynamics $\dot{\xi} = Dv_{s(t)}\xi$. Then $\xi(t)$ must stay on the same side of T. In particular, we can never have $\xi(t), \xi_{s(t)}, v_{s(t)}$ linearly dependent with $\lambda\, \xi(t) < 0$.



$\xi(t), \xi_{s(t)}, v_{s(t)}$ linearly independent in a regular energy level iff $\omega(\xi(t), \xi_{s(t)}) \neq 0$, for $\omega = dx \wedge dp_x + dy \wedge dp_y$ (take $\Omega = \frac{v^\flat}{|v|^2} \wedge \omega$).

So if $\exists t$ s.t. $\omega(\xi(t), \xi_{s(t)})$ changes sign and $\lambda\, \xi(t) < 0$ ("negative crossing") then $\nexists$ invariant torus through $s(0)$ transverse to $\mathcal{F}$.

In practice, reduce dimension of search space from 4 to 3 or 2 ...

Surface of section

Enough to test initial conditions on a transverse codimension-1 set Σ such that every bounded trajectory crosses it.

Every bounded trajectory comes to a local maximum of r, so take $\Sigma = \{s : p_r = 0, \dot{p}_r < 0\}$. For $\mu = 0$ this is $r > L^2$.

[Note additional non-existence criterion: if $s(t)$ never returns to Σ then it is not on an invariant torus (of any class).]

H conserved: $\Sigma_C = \Sigma \cap H^{-1}(-C/2)$. For $\mu = 0$, use coordinates (L, θ); allowed region $2L \leq C \leq 2L + L^{-2}$ (1 or 2 annuli).

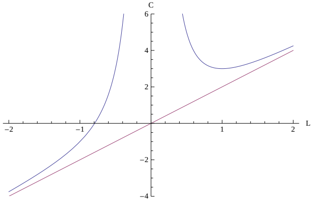


Figure: Allowed region in (L, C)

Symmetry planes

PCR3BP has time-reversal symmetry wrt $\theta \mapsto -\theta$, $p_r \mapsto -p_r$, $r \mapsto r$, $L \mapsto L$. Symmetry planes P_0, P_π: $p_r = 0$, $\theta = 0, \pi$

Every invariant torus transverse to $\mathcal{F}$ intersects P_0, P_π ($\geq$ twice). Maybe enough to look on them.

For μ small, expect tori destroyed by crossing orbit of secondary ($r > 1, L^2 < \frac{2r}{r+1}$, or v.v) or near resonances $\rho = N^3 \in \mathbb{Q}$.

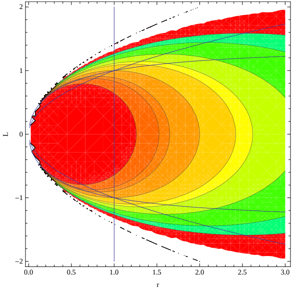


Figure: Curves delimiting crossing of $r = 1$, and some resonances, in a symmetry plane for $\mu = 0$; escape for $L^2 > 2r$; circular for $L^2 = r$

Numerics: Example orbit

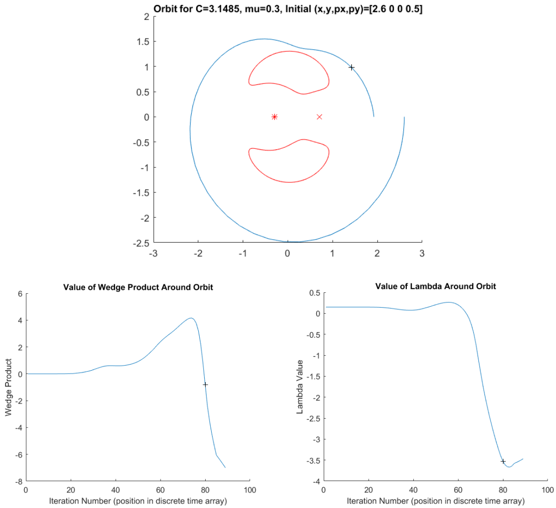


Figure: An orbit with a negative crossing

Symmetry plane P_0

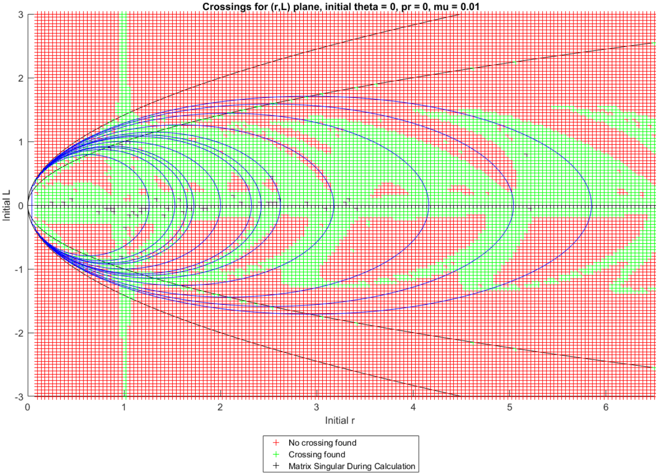


Figure: Results for $\mu = 0.01$ in symmetry plane $p_r = 0, \theta = 0$; superposed curves for $\mu = 0$ are $r = L^2$ for circular orbits, $2r = L^2$ for parabolic orbits, $(\rho^{-2/3}r - 1)^2 = 1 - \rho^{-2/3}L^2$ for $|\rho| = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, 2, 3, 4, 5$

Discussion

Dominant effect producing non-existence seems to be crossing orbit of secondary.

Foliation degenerates on the circular orbits, so expect non-existence near $r = L^2$, because even though there are invariant tori around the continuation of the circular orbits the thinnest ones are not transverse to the foliation.

Expect resonance to be important only for orbits that do not cross $r = 1$. This is $r > 1, \frac{2r}{r+1} < L^2 < 2r$, where Kepler ellipses remain in $r > 1$, or $r < 1, L^2 < \frac{2r}{r+1}$, where they remain in $r < 1$.

Even then, resonances might not show up for one or other choice of symmetry plane. Indeed, surface of section plot suggests that they do not show up in $\theta = 0$ and some are missing in $\theta = \pi$.

Surface of section plot

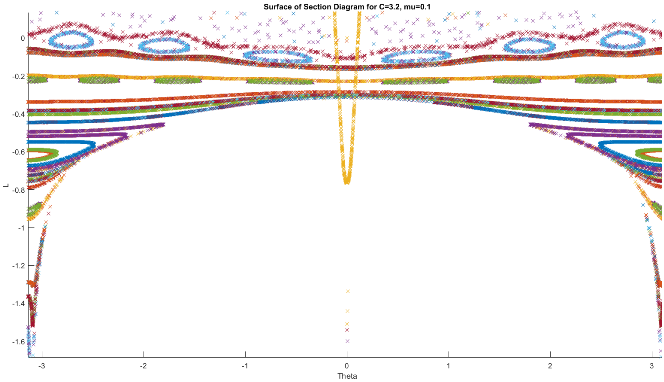


Figure: Surface of section plot $(p_r = 0, \dot{p}_r < 0)$ for $C = 3.2, \mu = 0.1$

Periodic orbits of Maslov index 0 do not experience negative crossings. So $\theta = 0$ is not a good choice for this method; $\theta = \pi$ is better but still intersects some periodic orbits of Maslov index 0.

Symmetry plane P_π

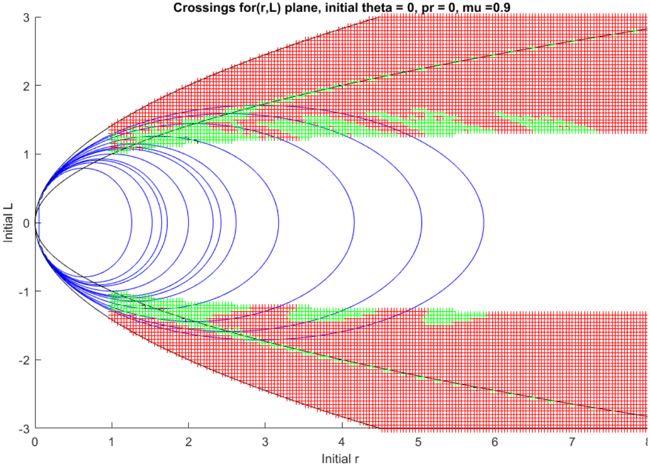
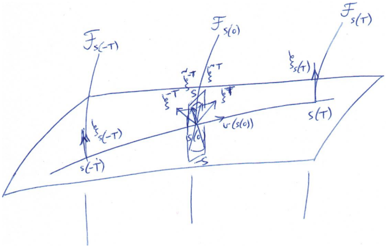


Figure: Plane $\theta = 0$ for $\mu = 0.9$, equivalent to $\theta = \pi$ for $\mu = 0.1$

So need a better symmetry plane (perhaps where return map = reflection) or "killends" extension of method.

Killends

Step 1: Integrate from $\xi_{s(t)}$ to vector ξ^t at $s(0)$ for $t \in [-T, T]$. Let $\tilde{\xi}^t$ be the quotient by $\mathbb{R}v(s(0))$, and S be the set of directions of $\tilde{\xi}^t, t \in [-T, T]$. If $\exists$ invariant torus $\mathcal{T}$ through $s(0)$ transverse to $\mathcal{F}$ then $S \cap (-S) = \emptyset$. So if non-empty there is no such torus.



Step 2: Take a transverse section Σ to v; do step 1 for all its points. Then tangent τ to $\Sigma \cap \mathcal{T}$ is in complement C to $S \cup (-S)$. If integrating the differential inclusion $\tau \in C$ in one direction from a point forces termination in the set where $C = \emptyset$ then C at that point can be set to empty (and at all points reached on the way). More efficient to work outwards from the set where $C = \emptyset$.

Illustration of killends for standard map

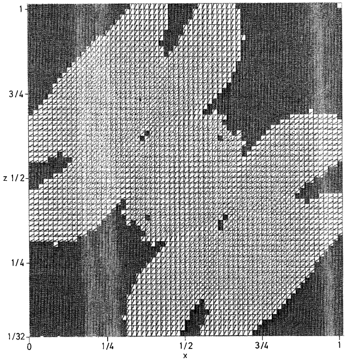


Fig. 6. Conefield for rotational invariant circles of the standard map at $k \sim 1.05$. No circles pass through the black regions

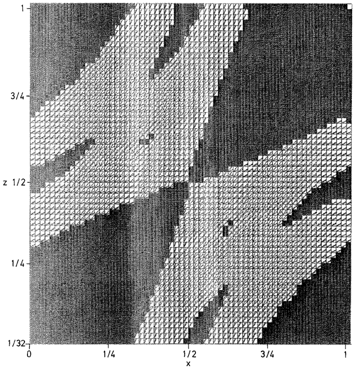


Fig. 7. Same as Fig. 6 after application of subsidiary criterion "killends"

Figure: From MacKay & Percival, Commun Math Phys 98 (1985) 469

Extensions

Better coordinates to plot symmetry planes for PCR3BP are $L' = \frac{L}{r^2}, r' = \frac{2r}{r+1}$, because non-escape region is $L' < 2$, circular orbits is $L' = 1$, and r' compactifies $(0, \infty)$ to $(0, 2)$ preserving 1.

Can apply to different foliations adapted to different classes of tori in different regions of phase space. Particularly interesting in the PCR3BP would be a foliation adapted to invariant tori surrounding the secondary mass, to determine sphere of influence.

Or even to a single foliation (with singularities) to cover all classes of tori, e.g. for pendulum could use gradient curves of H (has been applied by Duignan&Meiss to extend results of MacKay, Physica D 36 (1989) 64 for the two-wave Hamiltonian). Separate the classes using the singularities of the foliation. Best to regularise collisions with the primaries first.

All extends to more DoF for Lagrangian tori transverse to a Lagrangian foliation (cf. M,Meiss&Stark, Nonlin 2 (1989) 555, and correspondence with Robbins years ago about Maslov cycle).

3. Conclusion

Have presented the use of Hamiltonian dynamics to design magnetic fields in which guiding centre motion is integrable.

And a method to establish regions of phase space through which pass no invariant tori of given class.

If someone good at numerics for Hamiltonian systems would like to come and implement the non-existence criterion for FGCM in some example magnetic fields, let me know.