Monoclinal waves in granular flow: The interplay between nonlinearity and slow diffusion

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Based on...

J. Fluid Mech. (2018), *vol.* 843, *pp.* 810–846. © Cambridge University Press 2018 810
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The granular monoclinal wave

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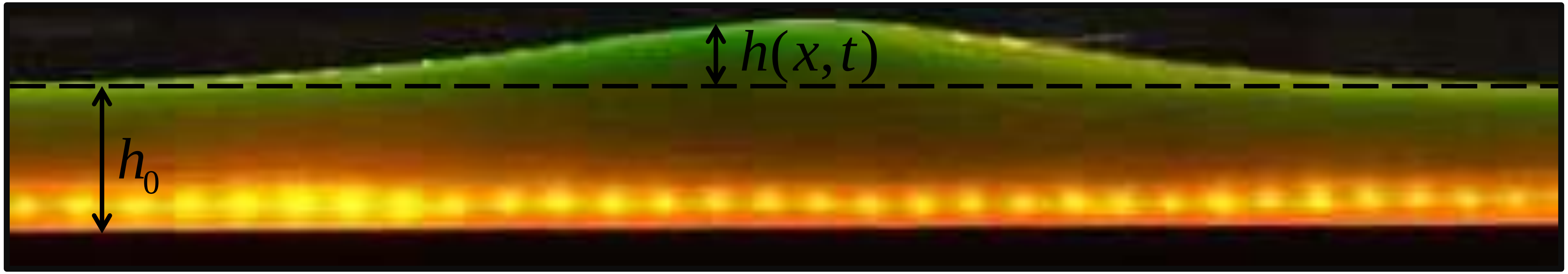
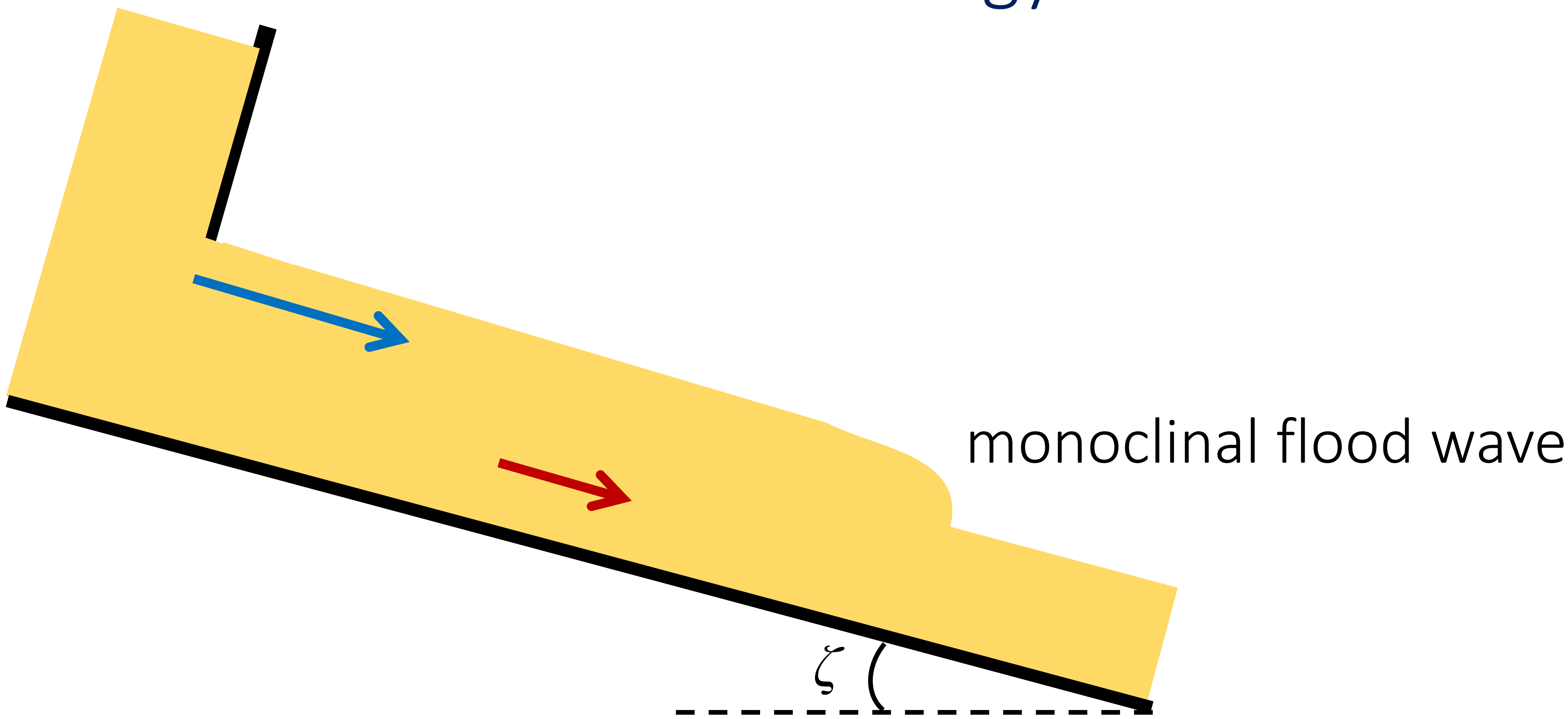
An iconic example: the KdV soliton

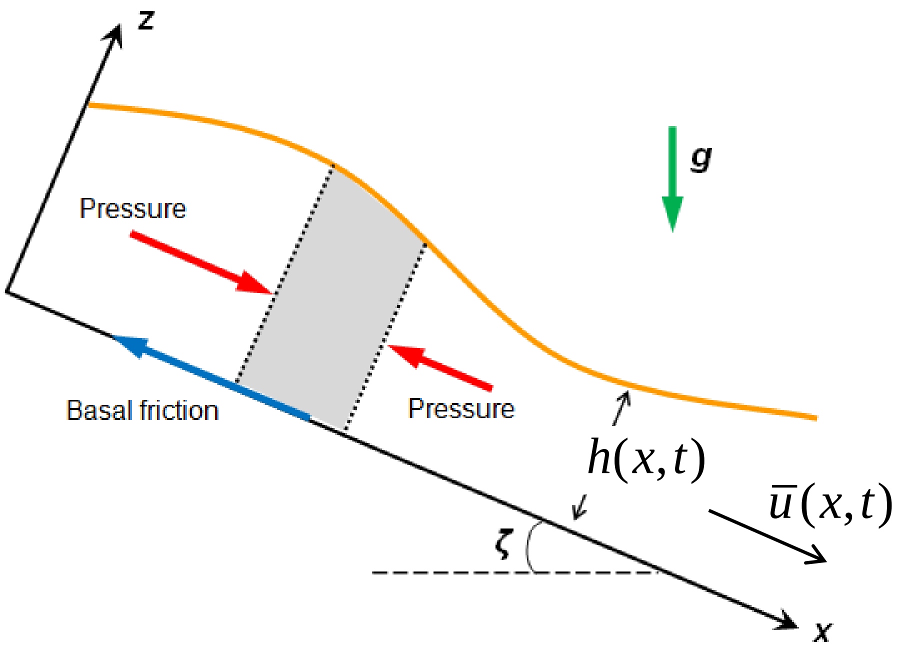
$$\frac{\partial h}{\partial t} + \sqrt{gh_0}\left(1 + \boxed{\frac{3h}{2h_0}\right)\frac{\partial h}{\partial x}} + \frac{1}{6}\sqrt{gh_0}\,h_0^2 \boxed{\frac{\partial^3 h}{\partial x^3}} = 0$$

A balance between: *non-linearity* & *dispersion*



Phenomenology





Mathematical model:
Saint-Venant equations

Continuity:

$$\frac{\partial h}{\partial t}+\frac{\partial\left(h\bar{u}\right)}{\partial x}=0$$

Momentum:

$$\frac{\partial}{\partial t}\left(h\bar{u}\right)+\frac{\partial}{\partial x}\left(h\bar{u}^2\right)=hg\sin\zeta-\frac{\partial}{\partial x}\left(\tfrac{1}{2}gh^2\cos\zeta\right)-\mu(h,\bar{u})hg\cos\zeta+\nu(\zeta)\frac{\partial}{\partial x}\left(h^{3/2}\frac{\partial\bar{u}}{\partial x}\right)$$

momentum change — *gravity* — *pressure gradient* — *basal friction* — *viscous force*

Rheological law 1: the friction coefficient $\mu(h,\bar{u})$

$$\mu(h,\bar{u}) = \tan\zeta_1 + \frac{\tan\zeta_2 - \tan\zeta_1}{1 + \dfrac{\beta\, h^{3/2}\sqrt{g\cos\zeta}}{\mathcal{L}\,\bar{u}}}$$

for $F > \beta$

with $$F = \frac{\bar{u}}{\sqrt{gh\cos\zeta}}$$: Froude number [Inertia/Gravity]

In our system:

$\zeta_1 = 32.9^\circ$

$\zeta = 33.3^\circ$

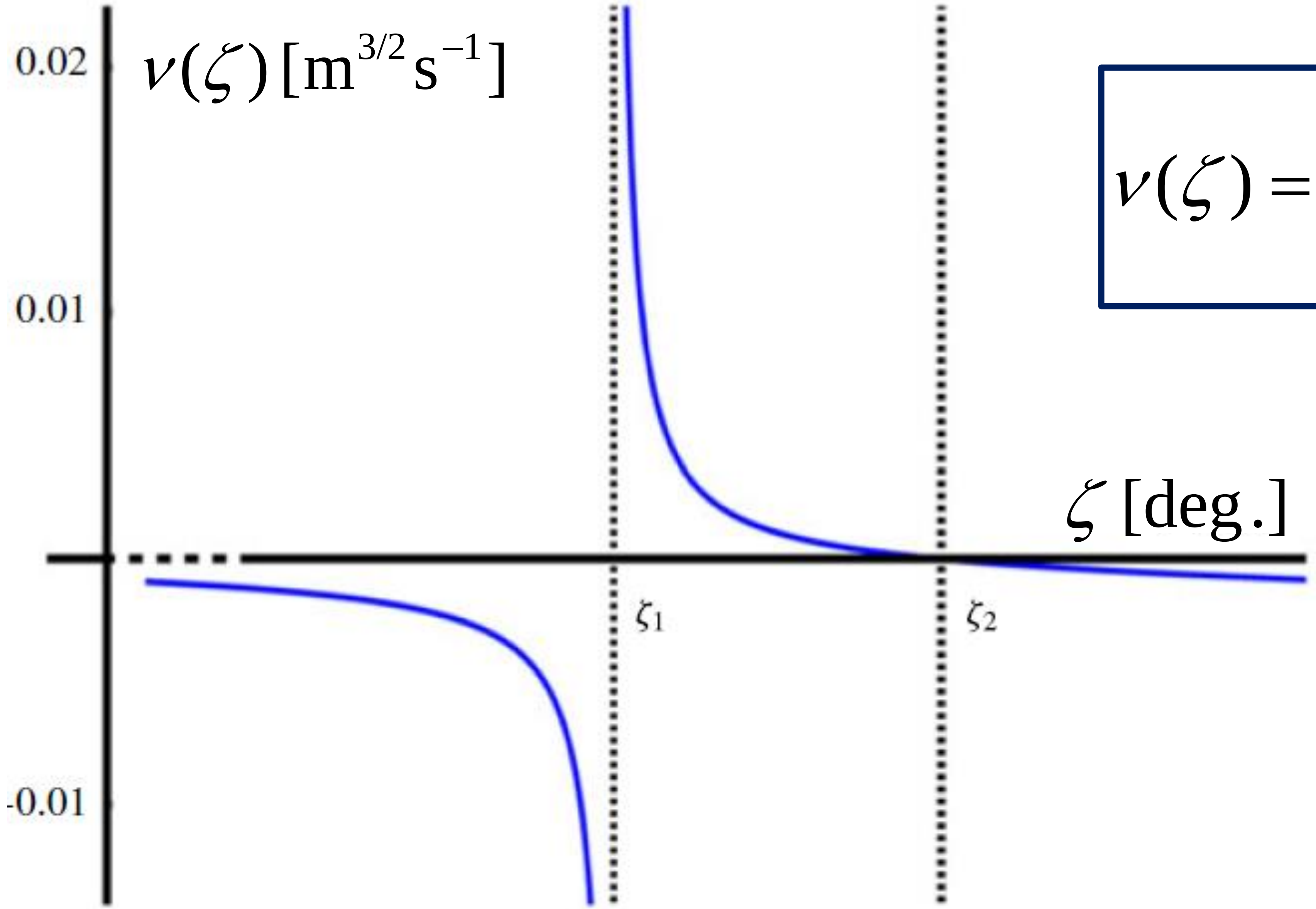
$\zeta_2 = 42.0^\circ$

$\beta = 0.5$

$\mathcal{L} = 1\text{mm}$

O. Pouliquen and Y. Forterre,
J. Fluid Mech. **453** (2002)

Rheological law 2: the viscosity coefficient $\nu(\zeta)$



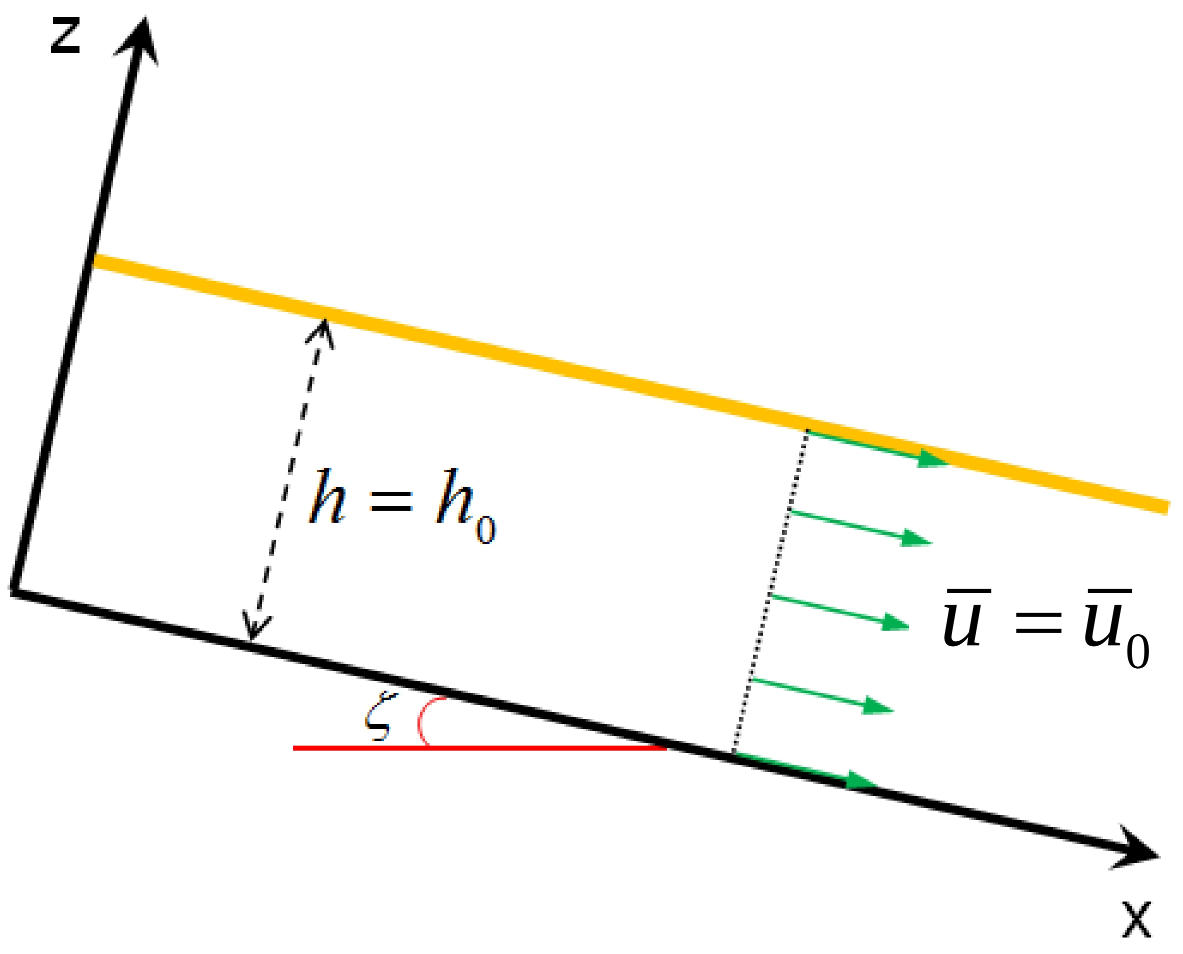
$$\nu(\zeta) = \frac{2\mathcal{L}\sqrt{g}\sin\zeta}{9\beta\sqrt{\cos\zeta}}\,\gamma(\zeta)$$

with

$$\gamma(\zeta) = \frac{\tan\zeta_2 - \tan\zeta}{\tan\zeta - \tan\zeta_1}$$

J. M. N. T. Gray and A. N. Edwards, *J. Fluid Mech.* **755** (2014)

Basic solution: steady uniform flow



Friction balances *Gravity*

$$\mu(h_0, \bar{u}_0) = \tan\zeta$$

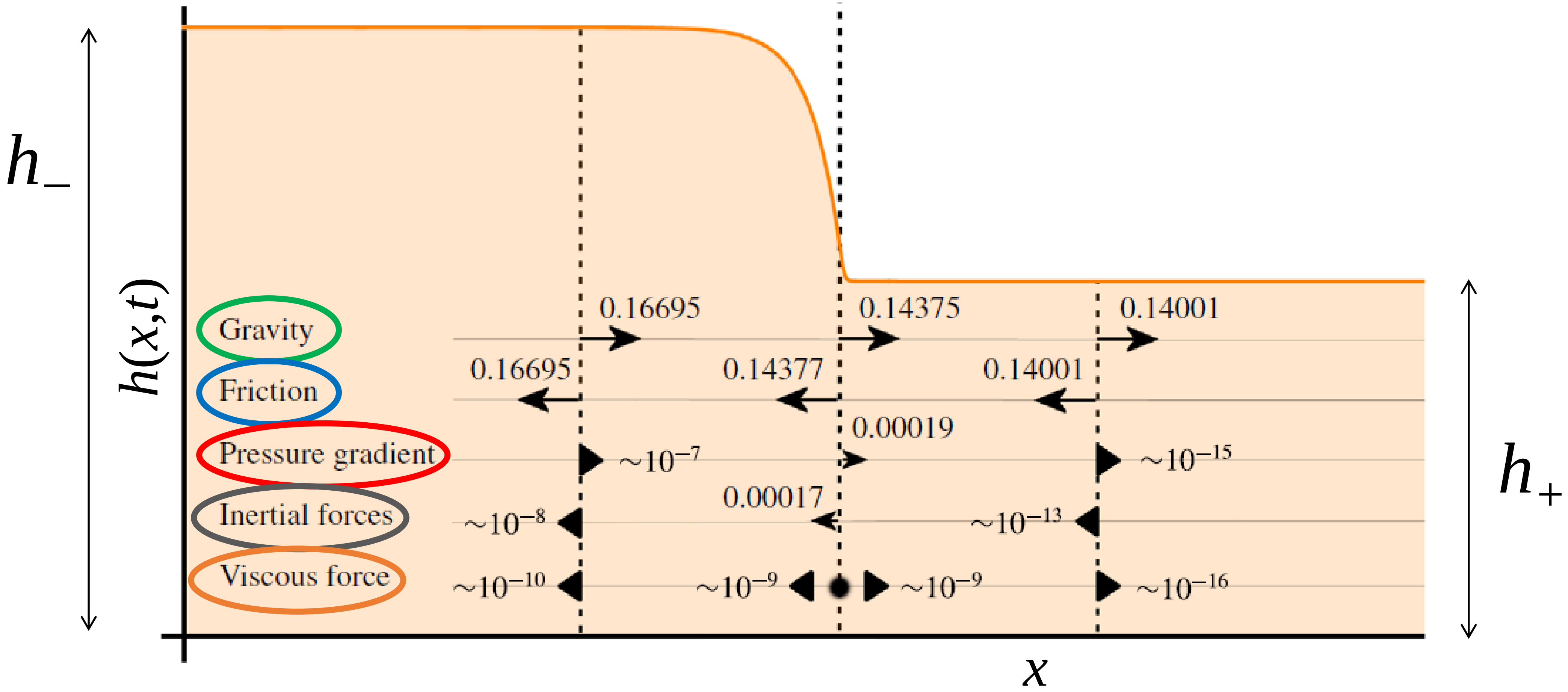
$$\bar{u}_0 = \frac{\beta\sqrt{g\cos\zeta}}{\mathcal{L}\,\gamma(\zeta)}\,h_0^{3/2}$$

Balance of forces

negligible

$$0 = -\frac{\partial}{\partial t}(h\bar{u}) - \frac{\partial}{\partial x}(h\bar{u}^2) + hg\sin\zeta - \frac{\partial}{\partial x}\left(\tfrac{1}{2}gh^2\cos\zeta\right) - \mu(h,\bar{u})hg\cos\zeta + \nu\frac{\partial}{\partial x}\left(h^{3/2}\frac{\partial\bar{u}}{\partial x}\right)$$

inertial forces *gravity* *pressure gradient* *friction* *viscous forces*



Inviscid limit ($\nu \longrightarrow 0$)

Continuity:

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0$$

Momentum:

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(h\bar{u}^2) = hg\sin\zeta - \frac{\partial}{\partial x}\left(\tfrac{1}{2}gh^2\cos\zeta\right) - \mu(h,\bar{u})hg\cos\zeta$$

Reduced momentum

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} = g\sin\zeta - g\cos\zeta\frac{\partial h}{\partial x} - \mu(h,\bar{u})g\cos\zeta$$

Rescalings:

$$h = h_\_ \tilde{h} \qquad x = h_\_ \tilde{x} \qquad \bar{u} = \bar{u}_\_ \tilde{u} \qquad t = (h_\_ / \bar{u}_\_)\tilde{t}$$

with

$$\bar{u}_\_ = \frac{\beta\sqrt{g\cos\zeta}}{\mathcal{L}\,\gamma(\zeta)} h_\_^{3/2}$$

The non-dimensional system

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \left(\tilde{h} \tilde{u} \right)}{\partial \tilde{x}} = 0$$

$$F_{\_}^{2} \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} \right) = -\frac{\partial \tilde{h}}{\partial \tilde{x}} + \tan \zeta - \tan \zeta_1 - \frac{\tan \zeta_2 - \tan \zeta_1}{1 + \gamma \tilde{h}^{3/2} / \tilde{u}}$$

for stable monoclinal waves: $\beta < F_{\_} < 2/3$

→ $\beta^2 < F_{\_}^2 < 0.44$

Creeping flow limit ($F_- << 1$)

$$F_-^2\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}\right)=-\frac{\partial\tilde{h}}{\partial\tilde{x}}+\tan\zeta-\tan\zeta_1-\frac{\tan\zeta_2-\tan\zeta_1}{1+\gamma(\zeta)\tilde{h}^{3/2}/\tilde{u}}$$

↓

$$0\approx-\frac{\partial\tilde{h}}{\partial\tilde{x}}+\tan\zeta-\tan\zeta_1-\frac{\tan\zeta_2-\tan\zeta_1}{1+\gamma(\zeta)\tilde{h}^{3/2}/\tilde{u}}$$

↓

$$\tilde{u}\approx\gamma(\zeta)\frac{\tan\zeta-\tan\zeta_1-(\partial\tilde{h}/\partial\tilde{x})}{\tan\zeta_2-\tan\zeta+(\partial\tilde{h}/\partial\tilde{x})}\tilde{h}^{3/2}$$

Creeping flow limit (continued)

$$\tilde{u} \approx \gamma(\zeta) \frac{\tan\zeta - \tan\zeta_1 - (\partial\tilde{h}/\partial\tilde{x})}{\tan\zeta_2 - \tan\zeta + (\partial\tilde{h}/\partial\tilde{x})} \tilde{h}^{3/2}$$

Taylor expansion

for a monoclinal wave: $\partial\tilde{h}/\partial\tilde{x} << 1$ → $\frac{a-\varepsilon}{b+\varepsilon} \approx \frac{a}{b} - \frac{a+b}{b^2}\varepsilon$

$$\tilde{u} \approx \tilde{h}^{3/2} - \frac{1+\gamma(\zeta)}{\tan\zeta_2 - \tan\zeta} \frac{\partial\tilde{h}}{\partial\tilde{x}} \tilde{h}^{3/2}$$

closure for the continuity equation!

nonlinearity vs. slow diffusion

Continuity:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \left(\tilde{h} \tilde{u} \right)}{\partial \tilde{x}} = 0$$

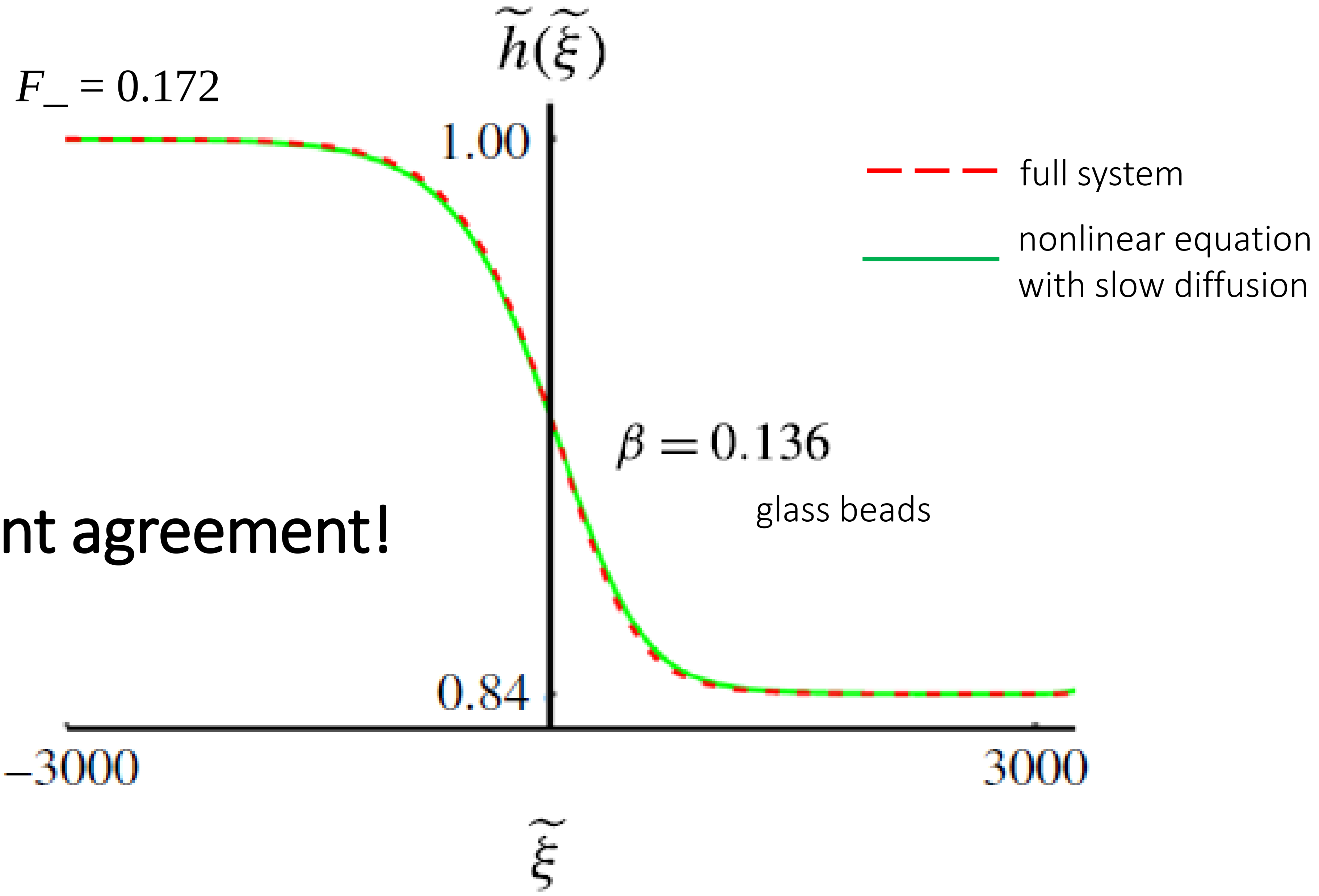
Closure:

$$\tilde{u} \approx \tilde{h}^{3/2} - \frac{1+\gamma(\zeta)}{\tan \zeta_2 - \tan \zeta} \frac{\partial \tilde{h}}{\partial \tilde{x}} \tilde{h}^{3/2}$$

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{5}{2} \tilde{h}^{3/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} = \frac{2}{7} \frac{1+\gamma(\zeta)}{\tan \zeta_2 - \tan \zeta} \frac{\partial^2 (\tilde{h}^{7/2})}{\partial \tilde{x}^2}$$

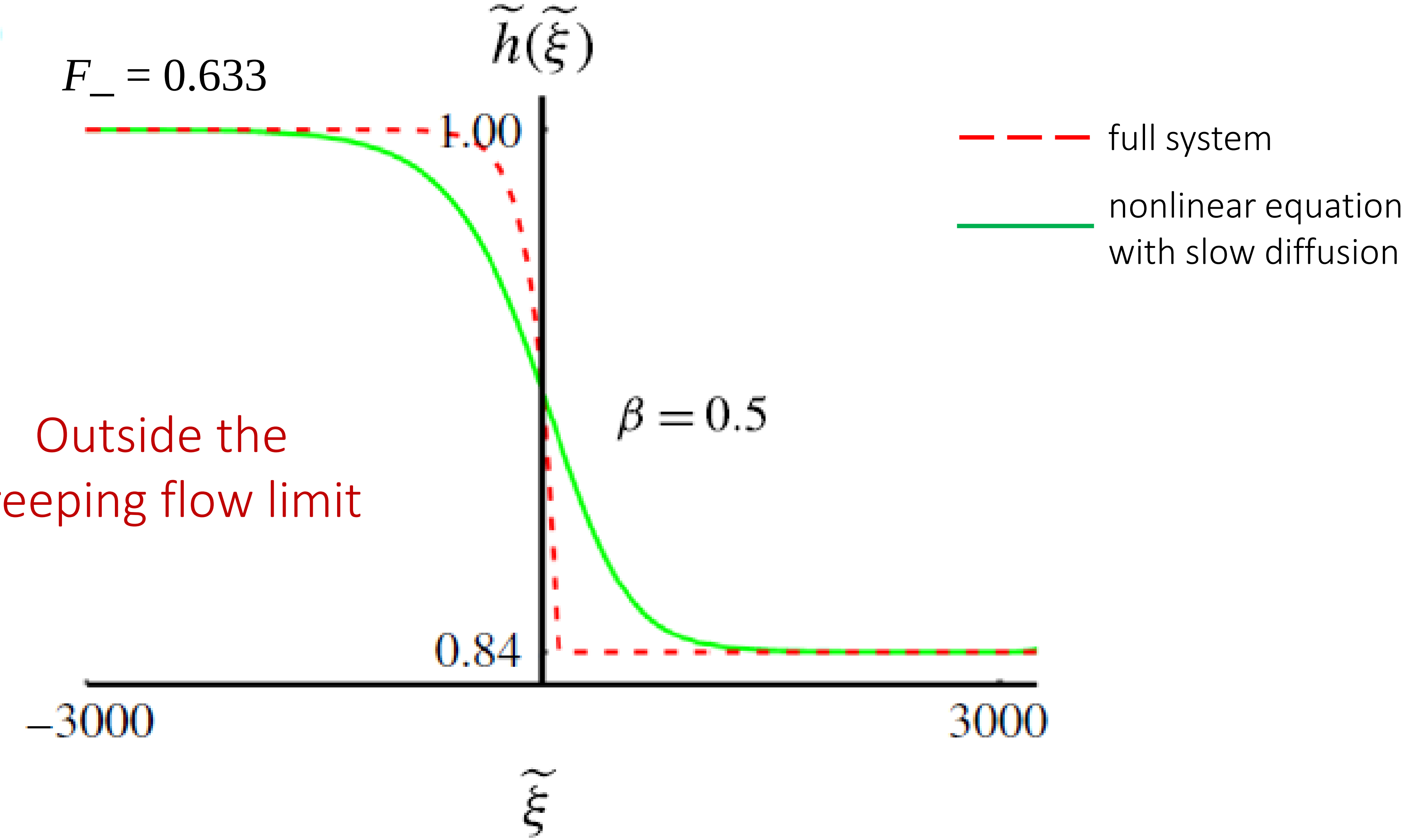
nonlinearity vs. slow diffusion

Numerical results



excellent agreement!

Numerical results (continued)



Conclusion

The shape of a granular monoclinal wave

in the creeping flow limit ($F << 1$)

is determined by the competition

of two opposing mechanisms:

nonlinearity and slow diffusion

Thank you !