Quantum Metrology with Continuous-variable Nonclassical States

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Outline

Continuous-variable (CV) Quantum Optical Field States

Quantum Optical Metrology: Precision on Phase Estimation

Engineering Nonclassical CV States for Quantum Metrolog

Summary and Conclusions

Coherent States

- Coherent States of light:
  The eigenstates of bosonic annihilation operator $\hat{a}$, or displaced vacuum state

$$\hat{a}|\gamma\rangle = \gamma|\gamma\rangle \quad \text{or} \quad |\gamma\rangle = \hat{D}(\gamma)|0\rangle; \qquad \hat{D}(\gamma) = e^{(\gamma\hat{a}^\dagger - \gamma^*\hat{a})},$$

- γ is complex number.
- In terms of coherent superposition of photon-number state basis:

$$|\gamma\rangle = e^{-\frac{|\gamma|^2}{2}} \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}} |n\rangle.$$

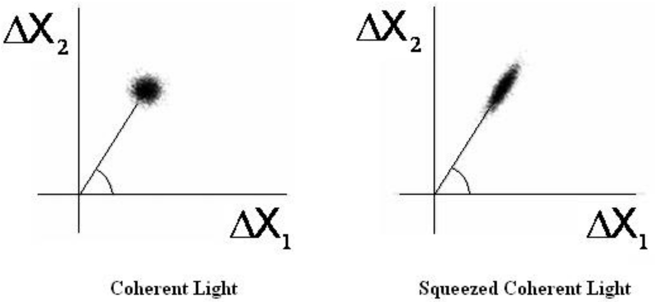
- The uncertainty product is minimum

$$(\Delta\hat{X}_1)^2 = \frac{1}{4} = (\Delta\hat{X}_2)^2 \qquad \Delta\hat{X}_1 \, \Delta\hat{X}_2 = \frac{1}{4}$$

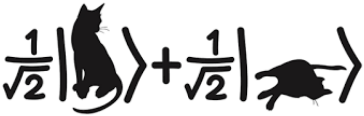
Squeezed States

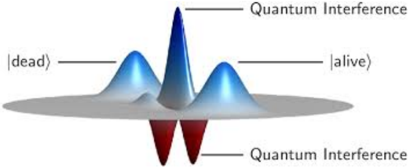
- Squeezed States

$$\hat{D}(\gamma)\hat{S}(\varsigma)|0\rangle = |\gamma, r\rangle; \qquad \hat{S}(\varsigma) = \exp[\frac{1}{2}(\varsigma^* \hat{a}^2 - \varsigma \hat{a}^{\dagger\,2})]$$



Schrödinger Cat States





- Schrödinger Cat States are the superposition of macroscopically distinct (180° out phase) coherent states[1]

$$|\gamma\rangle_{YS} = \frac{\mathcal{N}}{\sqrt{2}}\left(|\gamma\rangle + e^{i\pi/2}|-\gamma\rangle\right)$$

- The normalization constant is given as $\mathcal{N} = \left(1 + \exp(-2\gamma^2)\cos(\phi)\right)^{\frac{-1}{2}}$.

[1] D. Stoler, Phys. Rev. Lett. 57, 13 (1986)

Entangled Coherent States

- Entangled coherent states[2] are the correlated superpositions

$$|\Psi\rangle_{AB} = \mathcal{N}_{AB}\left[|\gamma_A\rangle \otimes |-\gamma_B\rangle + e^{i\phi}|-\gamma_A\rangle \otimes |\gamma_B\rangle\right]$$

- They can be obtained as

$$|\Psi\rangle_{AB} = \left[\hat{I}_A \otimes \hat{\Pi}_B + e^{i\phi}\hat{\Pi}_A \otimes \hat{I}_B\right]|\gamma_A\rangle \otimes |\gamma_B\rangle,$$

- Here $\hat{I}_A$ ($\hat{I}_B$) denotes the identity operator on subspace A (subspace B) and Π is photon number parity operator

$$\Pi|n\rangle = (-1)^{\hat{a}^\dagger \hat{a}}|n\rangle = (-1)^n|n\rangle$$

[2] B. Wielinga and B. C. Sanders J Mod. Opt., 40, 1923 (1993)

Classical vs Nonclassical Paradigms

Classical-like behavior of quantum optical coherent states

- Poisson photon counting statistics

$$P_n = |\langle n|\gamma\rangle|^2 = e^{-|\gamma|^2}\frac{|\gamma|^{2n}}{n!}, \quad \langle \hat{n}\rangle = \langle\gamma|\hat{a}^\dagger\hat{a}|\gamma\rangle = |\gamma|^2,$$

- Poisson statistic is a benchmark of classical behavior.
- Positive phase-space probability distribution (the Glauber-Sudarshan P function[3])

$$P(\gamma) = \frac{e^{|\gamma|^2}}{\pi^2}\int e^{|\alpha|^2}\langle-\alpha|\rho|\alpha\rangle e^{\alpha^*\gamma-\alpha\gamma^*}d^2\alpha$$

- For coherent states $P(\gamma) = \delta^2(\gamma-\beta)$.
- Stability of minimum uncertainty under temporal evolution.

[3] E. Sudarshan, PRL, 10(7):277, (1963)

Nonclassicality Criteria for CV optical field states

- Sub-Poisson Photon Statistics
  It is probed by the Mandel Q parameter [4]

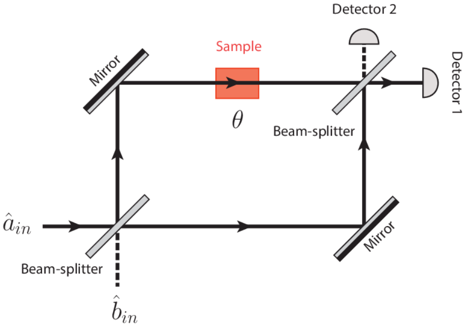
$$Q = \frac{(\Delta n)^2}{\langle n \rangle} - 1, \qquad (\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$$

- For $Q < 0$, the distribution is sub-Poissonian, otherwise, it is Poissonian for $Q = 0$ and super-Poissonian for $Q > 0$.
- Negative volume of phase-space
  measured by Glauber-Sudarshan P or Wigner function
- Quadrature Squeezing
  if either $(\Delta \hat{X}_i)^2 < \frac{1}{4}$ with $i = 1, 2$, provided $\Delta \hat{X}_1 \, \Delta \hat{X}_2 = \frac{1}{4}$

[4] L. Mandel, Optics Letters, 4 (7):205, (1979)

Outline

Transformation of Quantized Field by Interferometer



- Beam Splitter transformation

$$\hat{a}_{out} = \frac{1}{\sqrt{2}}\left(\hat{a}_{in} + i\hat{b}_{in}\right), \qquad \hat{b}_{out} = \frac{1}{\sqrt{2}}\left(\hat{b}_{in} + i\hat{a}_{in}\right)$$

- Phase shifter transformation

$$\hat{a} \to \sqcap_p^\dagger(\theta)\hat{a}\sqcap_p(\theta) = \hat{a}e^{-i\theta}, \qquad \sqcap_p(\theta) = \exp\left(-i\hat{a}^\dagger\hat{a}\theta\right)$$

Transformation of Quantized Field by Interferometer

- Transformation by Interferometer:

$$\hat{a}_{out}(D_1) = \frac{1}{\sqrt{2}}\left(-\frac{\hat{a}+i\hat{b}}{\sqrt{2}} - ie^{-i\theta}\frac{\hat{a}+i\hat{b}}{\sqrt{2}}\right),$$

$$\hat{a}_{out}(D_1) = \frac{1}{\sqrt{2}}\left(-\frac{\hat{a}+i\hat{b}}{\sqrt{2}} - ie^{-i\theta}\frac{\hat{a}+i\hat{b}}{\sqrt{2}}\right),$$

- If the field b_{in} is in the vacuum state:

$$\langle \hat{a}^\dagger_{out}\hat{a}_{out}\rangle = \sin^2\frac{\theta}{2}\langle \hat{a}^\dagger_{in}\hat{a}_{in}\rangle \qquad \langle \hat{b}^\dagger_{out}\hat{b}_{out}\rangle = \cos^2\frac{\theta}{2}\langle \hat{a}^\dagger_{in}\hat{a}_{in}\rangle$$

Sensitivity of an Optical Interferometer and Bounds Phase Estimation

- The sensitivity of an optical interferometer[5] is defined as

$$\Delta\theta = \frac{\Delta S}{|\partial S / \partial\theta|}$$

  Where S is the detected output signal and ΔS the fluctuation in the signal.
- Thus above defined sensitivity depends on:
  - the input states to the interferometer
  - the detection scheme, and
  - quantitative measures used to characterize the sensitivity of the interferometer

[5] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A 33, 4033 (1986)

Heisenber-Limited Interferometery

- For coherent state input, the measure output signal is

$$S = \sin^2 \frac{\theta}{2} |\gamma|^2 = N_1; \quad N_1 \text{ is average detected field.}$$

- Photon counting probability for coherent state is Poissonian,

$$\text{Therefore,} \quad \Delta N_1 = \sqrt{N_1}$$

- Thus the phase sensitivity comes out

$$\Delta\theta \approx \frac{1}{|\gamma|} . \frac{1}{|\cos\theta/2|} \approx \frac{1}{|\gamma|} \quad \text{for} \quad \theta \approx 0$$

- The number-phase Heisenberg uncertainty relation:

$$\Delta\theta\Delta N \geq 1 \longrightarrow \Delta\theta \geq \frac{1}{N}, \quad \text{Heisenberg Limit on phase estimation}$$

Cramer-Rao bound (CRB) on phase sensitivity

- CRB is defined in terms of Fisher information which is given as

$$F(\theta) = \sum_{n_1,n_2} \frac{1}{p(n_1 n_2)} \left(\frac{\partial p(n_1 n_2)}{\partial \theta} \right)^2$$

- $p(n_1 n_2)$ is the joint probability of detection n_1 (n_2) photons at port 1 (2).
- The largest phase sensitivity is given by CRB which is defined as [6]

$$(\Delta\theta)_{CRB} = \frac{1}{\sqrt{pF(\theta)}}$$

- p being the number of measurements done to estimate the phase θ.

[6] C.W. Helstrom, Quantum Detection and Estimation Theory (New York: Academic Press, 1976)

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The $SU(1,1)$ Lie algebra and relevant UIRs

- Set of generators, $\{\hat{\Lambda}_1, \hat{\Lambda}_2, \hat{\Lambda}_0\}$, span the $su(1,1)$ Lie algebra, satisfying

$$[\hat{\Lambda}_1, \hat{\Lambda}_2] = -i\hat{\Lambda}_0, \qquad [\hat{\Lambda}_2, \hat{\Lambda}_0] = i\hat{\Lambda}_1, \qquad [\hat{\Lambda}_0, \hat{\Lambda}_1] = i\hat{\Lambda}_2$$

$$[\hat{\Lambda}_0, \hat{\Lambda}_\pm] = \pm\hat{\Lambda}_\pm, \quad [\hat{\Lambda}_-, \hat{\Lambda}_+] = 2\hat{\Lambda}_0 \text{ where } \hat{\Lambda}_\pm = \hat{\Lambda}_1 \pm i\,\hat{\Lambda}_2$$

- The relevant UIRs are given by positive discrete series $\mathcal{D}^k : \{|k,n\rangle, k > 0, n = 0,1,2,...\}$ satisfying the eigenvalue equations

$$\begin{aligned}
\hat{\mathcal{C}}|k,n\rangle &= k(k-1)|k,n\rangle, \quad \hat{\Lambda}_0|k,n\rangle = (n+k)|k,n\rangle, \\
\hat{\Lambda}_-|k,n\rangle &= \sqrt{n(2k+n-1)}\,|k,n-1\rangle, \\
\hat{\Lambda}_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}\,|k,n+1\rangle.
\end{aligned}$$

where $\hat{\mathcal{C}} = \hat{\Lambda}_0^2 - \frac{1}{2}(\hat{\Lambda}_+\hat{\Lambda}_- + \hat{\Lambda}_+\hat{\Lambda}_-)$ and k is Bargmann index.

The $SU(1,1)$ Coherent States of Light

- The eigenstates of $su(1,1)$ lowering operator $\hat{\Lambda}_-$

$$\Lambda_-|z,k\rangle = z|z,k\rangle, \quad z \text{ is a complex parameter}$$

- In terms of the basis $|n,k\rangle$:

$$|z,k\rangle = \frac{z^{k-\frac{1}{2}}}{\sqrt{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n+2k)}} |n,k\rangle$$

- Single-mode biphotonic realization $su(1,1)$ algebra

$$\hat{\Lambda}_+ = \frac{\hat{a}^{\dagger 2}}{2}, \qquad \hat{\Lambda}_- = \frac{\hat{a}^2}{2}, \qquad \hat{\Lambda}_0 = \frac{1}{4}(2\hat{a}^\dagger \hat{a} + 1)$$

- Bergmann index takes values $k = 1/4, 3/4$ and UIR maps onto photon-number states

$$|n\rangle \leftrightarrow |k,\bar{n}\rangle \quad \text{for} \quad n = 2(\bar{n}+k) - 1/2.$$

Excited (photon-added) $SU(1,1)$ Coherent States

- The excited coherent states are obtained as [7]

$$|z,k,m\rangle = (\hat{\Lambda}_+)^m |z,k\rangle.$$

- Expanding in terms of UIRs:

$$|z,k,m\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{z^n \sqrt{(n+m)!\ \Gamma(2k+n+m)\ \Gamma(2k)}}{n!\ \Gamma(2k+n)} |n+m,k\rangle$$

where the normalization constant is

$$\mathcal{N} = \Big[\sum_{n=0}^{\infty} \frac{|z|^{2n}(n+m)!\ \Gamma(2k+n+m)\ \Gamma(2k)}{[n!\ \Gamma(2k+n)]^2} \Big]^{\frac{-1}{2}}.$$

[7] H. B. Monir, N. Amir and S. Iqbal, Int. J. of Theor. Phys. 58 (2019) 1776

Photon Counting Statistics of excited $SU(1,1)$ CSs

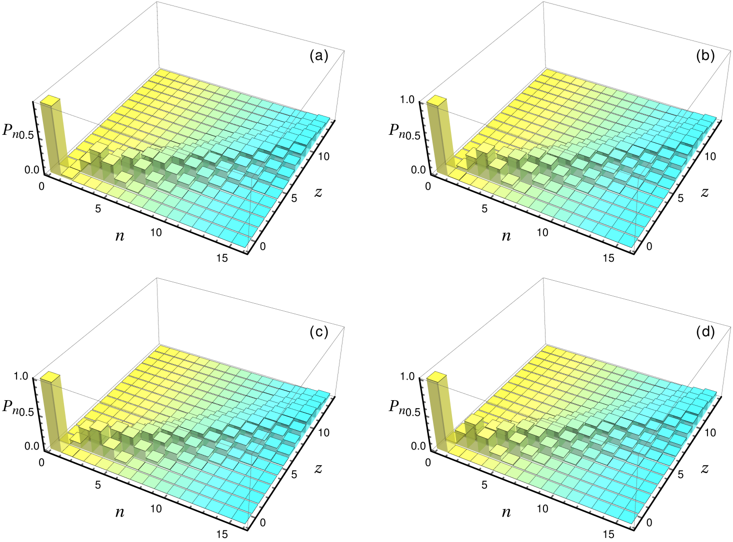

Figure: For $m = 8$ for (a)$k = \frac{1}{4}$ (b) $k = \frac{1}{2}$ (c) $k = \frac{3}{4}$ and (d) $k = 1.5$

Sub-Poissonian Photon Counting Probability Distribution

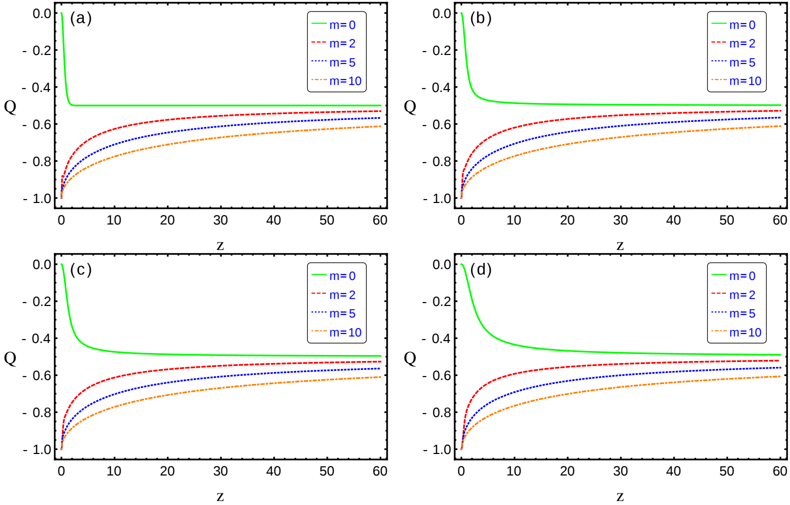

Figure: Mandel Q Parameter for (a)$k = \frac{1}{4}$ (b) $k = \frac{1}{2}$ (c) $k = \frac{3}{4}$ and (d) $k = 1.5$

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- First the properties of continuous-variable optical field states have been reviewed in the context of optical interferometry.
- Then we discussed the phase sensitivity of an optical interferometer and various bounds on the precision phase estimation, such as, Heisenberg Limit and Cramer-Rao bound (CRB).
- The procedure of engineering of nonclassical states of light have been discussed which may help beating the standard quantum bounds on precision phase estimation.

Thanks

for your attention