Chaotic behavior of multidimensional Hamiltonian systems

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Outline

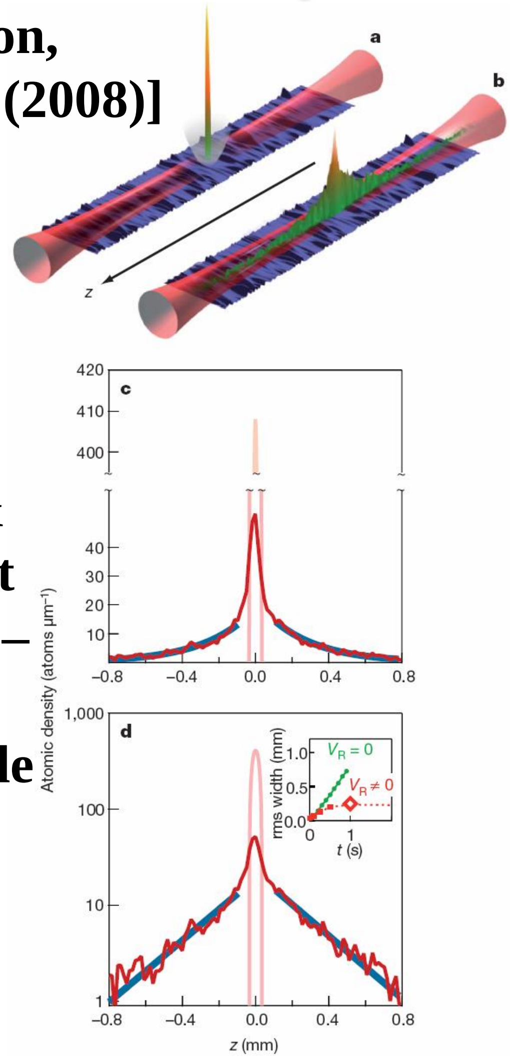
- **Brief overview of the dynamics of 1D Disordered lattices:**
  - ✓ **The quartic disordered Klein-Gordon (DKG) model**
  - ✓ **The disordered discrete nonlinear Schrödinger equation (DDNLS)**
  - ✓ **Different dynamical regimes**
- **Chaotic behavior of the DKG and DDNLS models in 1 and 2 spatial dimensions**
  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions (DVDs)**
- **Summary**

Work in collaboration with

Bob Senyange (PhD student):
1D and 2D DKG models

Bertin Many Manda (PhD student):
1D and 2D DDNLS models

Interplay of disorder and nonlinearity

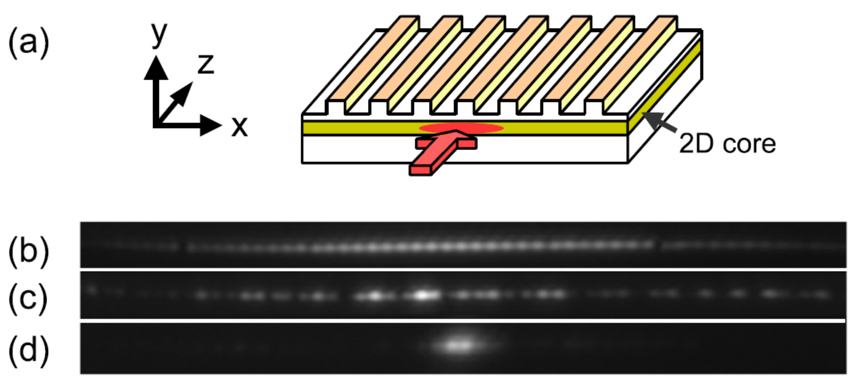


Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]

Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The 1D disordered Klein – Gordon model (1D DKG)

$$H_{1K}=\sum_{l=1}^{N}\frac{p_l^2}{2}+\frac{\tilde{\varepsilon}_l}{2}u_l^2+\frac{1}{4}u_l^4+\frac{1}{2W}\left(u_{l+1}-u_l\right)^2$$

with **fixed boundary conditions** $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: **W** and the **total energy E.** $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2},\frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l=A_l\exp(i\omega t)$. **Normal modes (NMs)** $A_{\nu,l}$ **- Eigenvalue problem:**

$$\lambda A_l=\varepsilon_l A_l-(A_{l+1}+A_{l-1})\ \text{with}\ \lambda=W\omega^2-W-2,\quad \varepsilon_l=W(\tilde{\varepsilon}_l-1)$$

The 1D disordered discrete nonlinear Schrödinger equation (1D DDNLS)

We also consider the system:

$$H_{1D}=\sum_{l=1}^{N}\varepsilon_l\left|\psi_l\right|^2+\frac{\beta}{2}\left|\psi_l\right|^4-\left(\psi_{l+1}\psi_l^*+\psi_{l+1}^*\psi_l\right)$$

where ε_l **chosen uniformly from** $\left[-\frac{W}{2},\frac{W}{2}\right]$ and β **is the nonlinear parameter.**

Conserved quantities: The energy and the norm $S=\sum_l\left|\psi_l\right|^2$ of the wave packet.

Distribution characterization (1D case)

We consider normalized energy distributions $\xi_l \equiv \frac{E_l}{\sum_m E_m}$

with $E_l = \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{4W}\left(u_{l+1} - u_l\right)^2$ for the DKG model,

and norm distributions $\xi_l \equiv \frac{|\psi_l|^2}{\sum_m |\psi_m|^2}$ for the DDNLS system.

Second moment: $m_2 = \sum_{l=1}^{N} \left(l - \bar{l}\right)^2 \xi_l$ with $\bar{l} = \sum_{l=1}^{N} l \xi_l$

Participation number: $P = \frac{1}{\sum_{l=1}^{N} \xi_l^2}$

measures the number of stronger excited sites in ξ_l.

Single site $P=1$. Equipartition of energy $P=N$.

Different dynamical regimes (1D case)

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes,

δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \propto t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \propto t^{1/2} \longrightarrow m_2 \propto t^{1/3}$

Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

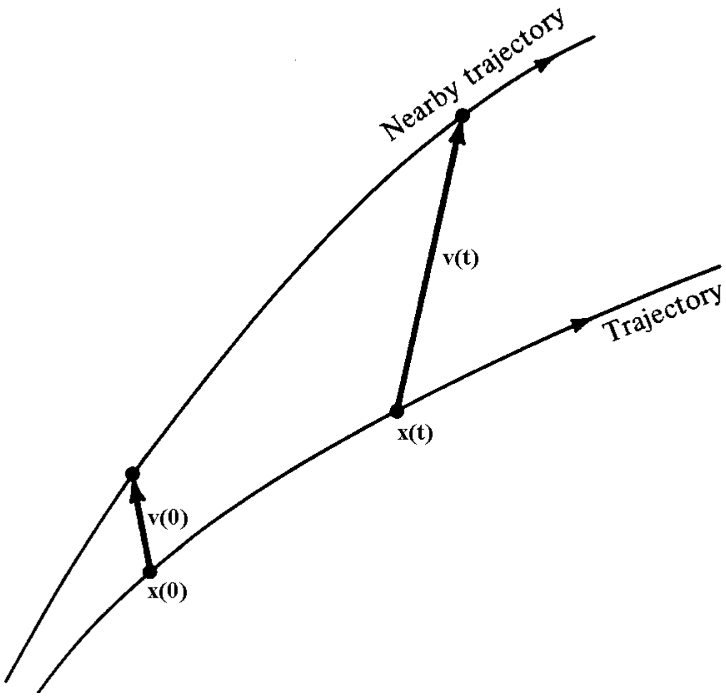
Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Variational Equations

We use the notation $\mathbf{x} = (u_1,u_2,\ldots,u_N,p_1,p_2,\ldots,p_N)^T$. The deviation vector from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2,\ldots,\delta x_{2N})^T$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called variational equations:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J}\cdot\mathbf{P}\cdot\mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_N & -\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{0}_N \end{pmatrix}, \quad \mathbf{P}_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i, j = 1, 2, \ldots, n$$

Maximum Lyapunov Exponent

Chaos: sensitive dependence on initial conditions.

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\text{mLCE} = \lambda_1 = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\lambda_1 = 0 \rightarrow$ **Regular motion**

$\lambda_1 > 0 \rightarrow$ **Chaotic motion**

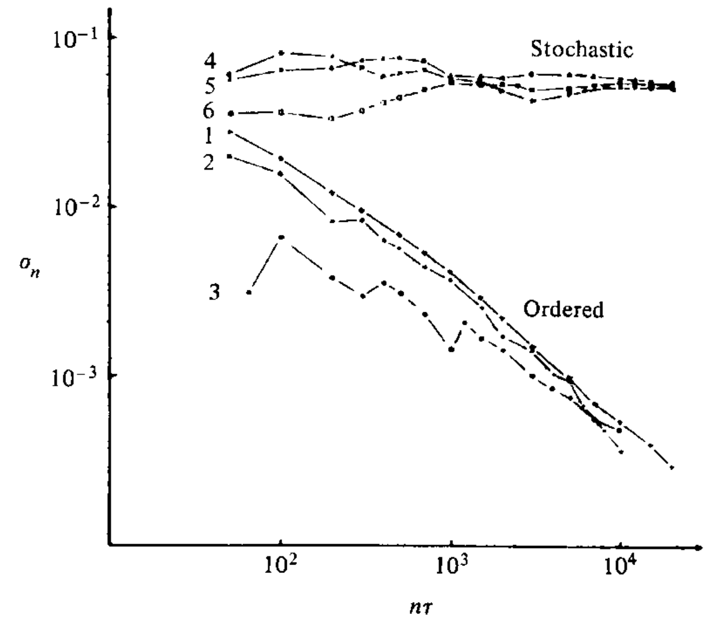


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

Symplectic integration (1D case)

We apply the 2-part splitting integrator ABA864 [Blanes et al., Appl. Num. Math. (2013) – Senyange & S., EPJ ST (2018)] to the DKG model:

$$H_{1K} = \sum_{l=1}^{N}\left(\frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2}u_l^2 + \frac{1}{4}u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2\right)$$

and the 3-part splitting integrator $ABC^6_{[SS]}$ [S. et al., Phys. Let. A (2014) – Gerlach et al., EPJ ST (2016) – Danieli et al., MinE (2019)] to the DDNLS system:

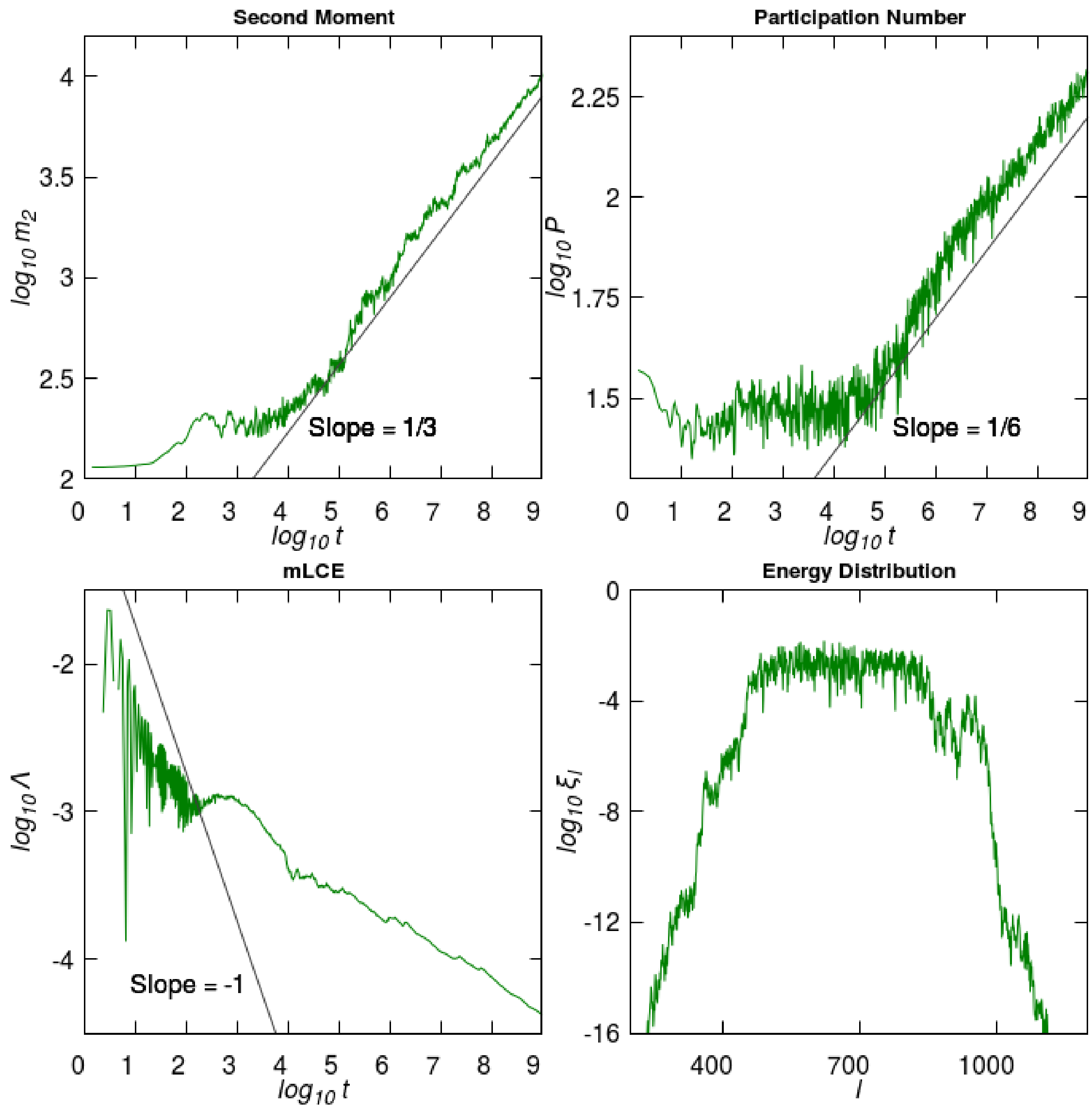
$$H_{1D} = \sum_l \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2}\left|\psi_l\right|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right), \quad \psi_l = \frac{1}{\sqrt{2}}\left(u_l + ip_l\right)$$

$$H_{1D} = \sum_l \left(\frac{\varepsilon_l}{2}\left(u_l^2 + p_l^2\right) + \frac{\beta}{8}\left(u_l^2 + p_l^2\right)^2 - u_n u_{n+1} - p_n p_{n+1}\right)$$

By using the so-called Tangent Map method we extend these symplectic integration schemes in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].

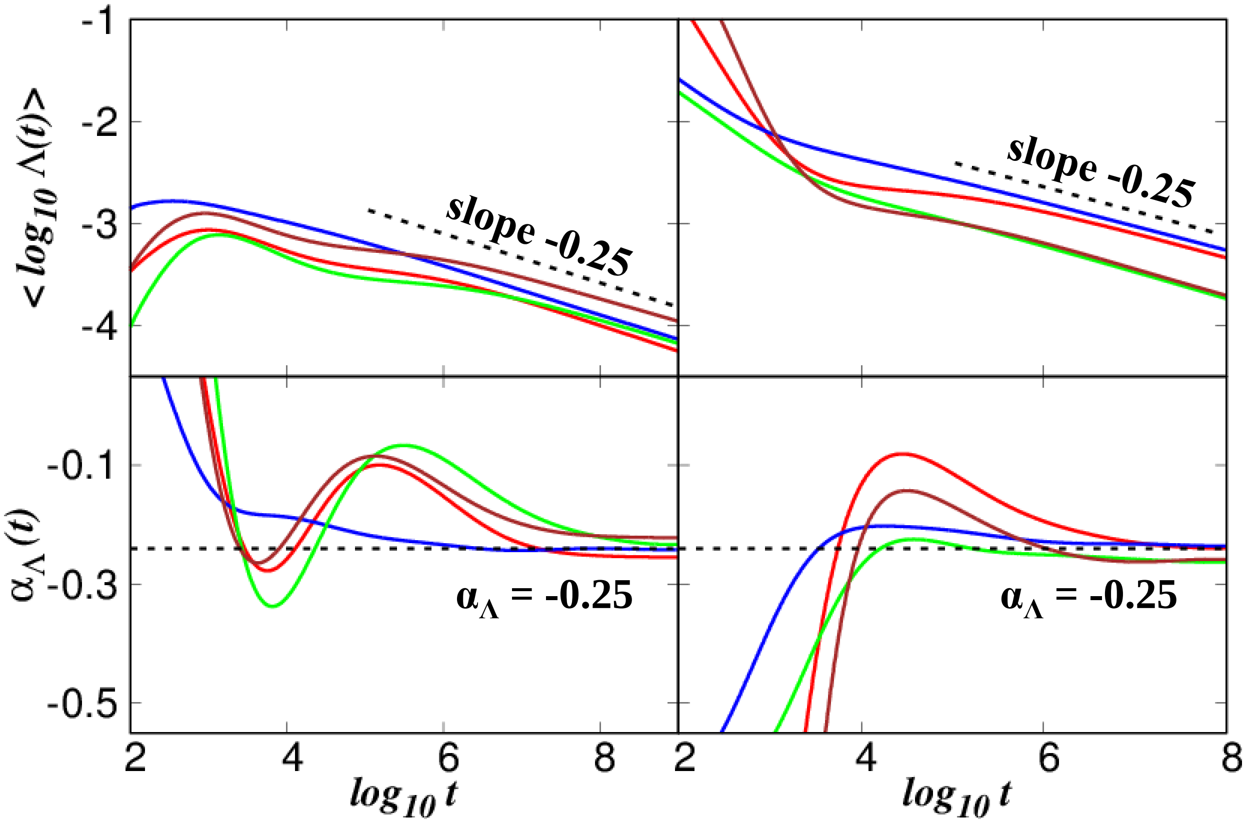
1D DKG: Weak Chaos

Block excitation
L=37 sites,
H_{1K}=0.37, W=3



Weak Chaos: 1D DKG and 1D DDNLS

1D DKG



1D DDNLS

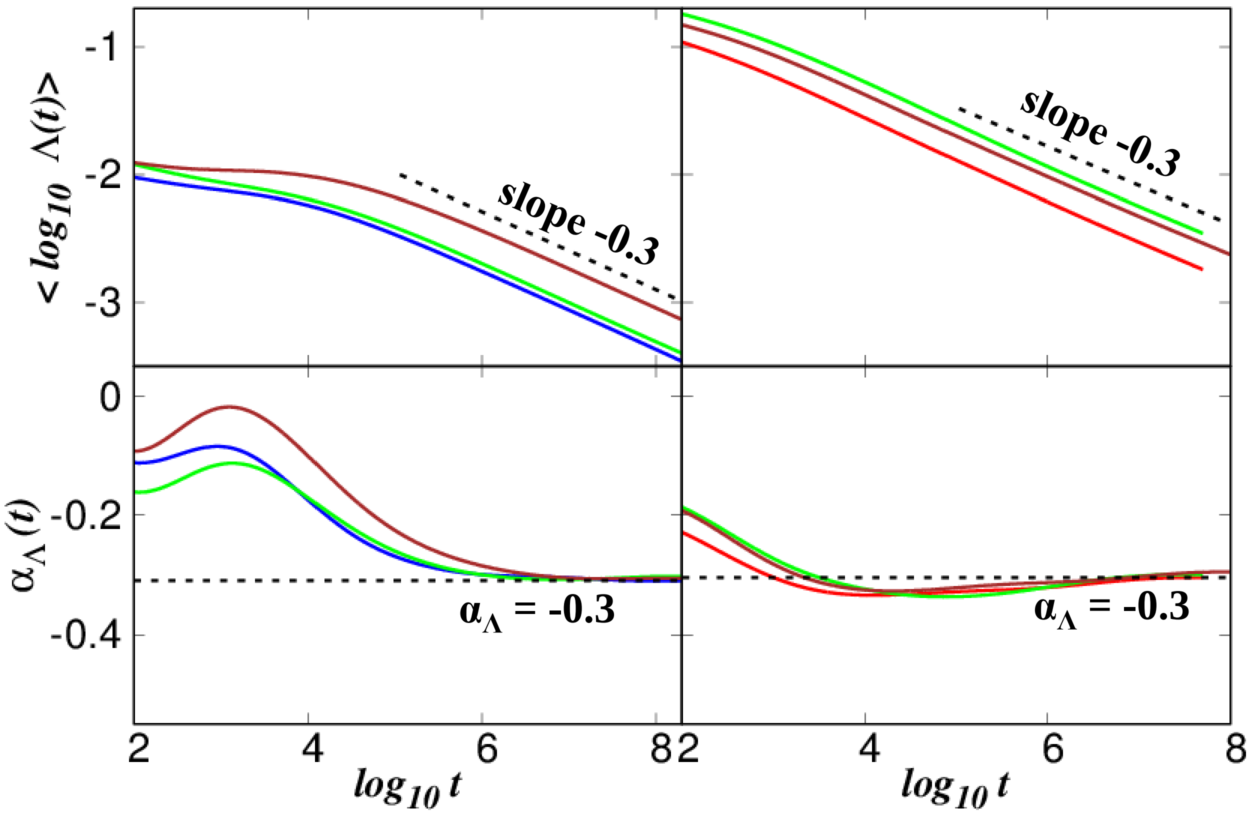
Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]

Block excitation (L=37 sites) H_{1K}=0.37, W=3	Block excitation (L=21 sites) β=0.04, W=4
Single site excitation H_{1K}=0.4, W=4	Single site excitation β=1, W=4
Block excitation (L=21 sites) H_{1K}=0.21, W=4	Single site excitation β=0.6, W=3
Block excitation (L=13 sites) H_{1K}=0.26, W=5	Block excitation (L=21 sites) β=0.03, W=3

1D DKG model also studied in S. et al., PRL (2013)

Strong Chaos: 1D DKG and 1D DDNLS

1D DKG



1D DDNLS

Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]

Block excitation (L=83 sites) H_{1K}=0.83, W=2
Block excitation (L=37 sites) H_{1K}=0.37, W=3
Block excitation (L=83 sites) H_{1K}=0.83, W=3

Block excitation (L=21 sites) β=0.62, W=3.5
Block excitation (L=21 sites) β=0.5, W=3
Block excitation (L=21 sites) β=0.72, W=3.5

The 2D DKG model

$$H_{2K} = \sum_{l,m} \left\{ \frac{p_{l,m}^2}{2} + \frac{\tilde{\varepsilon}_{l,m}}{2} u_{l,m}^2 + \frac{u_{l,m}^4}{4} + \right.$$
$$\left. + \frac{1}{2W} \left[\left(u_{l,m+1} - u_{l,m} \right)^2 + \left(u_{l+1,m} - u_{l,m} \right)^2 \right] \right\}$$

Again we have **fixed boundary conditions** and $\tilde{\varepsilon}_{l,m}$ **are chosen uniformly in** $\left[\frac{1}{2}, \frac{3}{2}\right]$.

The 2D DDNLS system

$$H_{2D} = \sum_{l,m} \left\{ \frac{\varepsilon_{l,m}}{2} \left(u_{l,m}^2 + p_{l,m}^2 \right) + \frac{\beta}{8} \left(u_{l,m}^2 + p_{l,m}^2 \right)^2 - \right.$$
$$\left. - \left(u_{l,m+1} u_{l,m} + u_{l+1,m} u_{l,m} + p_{l,m+1} p_{l,m} + p_{l+1,m} p_{l,m} \right) \right\}$$

Again ε_l **are chosen uniformly from** $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and β **is the nonlinear parameter.**

Conserved quantities: The energy H_{2D} and the norm $S = \sum_{l,m} \frac{u_{l,m}^2 + p_{l,m}^2}{2}$

Distribution characterization (2D case)

DKG: **energy distributions** $\xi_{l,m} = E_{l,m} / H_{2K}$ **with**

$$E_{l,m} = \frac{p_{l,m}^2}{2} + \frac{\tilde{\varepsilon}_{l,m} u_{l,m}^2}{2} + \frac{u_{l,m}^4}{4} + \frac{\left[\left(u_{l,m} - u_{l-1,m}\right)^2 + \left(u_{l,m} - u_{l,m-1}\right)^2 + \left(u_{l,m+1} - u_{l,m}\right)^2 + \left(u_{l+1,m} - u_{l,m}\right)^2\right]}{4W}$$

DDNLS: **norm distributions** $\xi_{l,m} = \left(u_{l,m}^2 + p_{l,m}^2\right) / 2S = s_{l,m} / S$

Second moment: $m_2 = \sum_{l,m} \left\| r_{l,m} - \bar{r} \right\|^2 \xi_{l,m}$ **with** $r_{l,m} = (l, m)$

and $\bar{r} = \left(\bar{l}, \bar{m}\right) = \left(\sum_{l,m} l \xi_{l,m} , \sum_{l,m} m \xi_{l,m} \right)$

Theoretical predictions [Flach et al., PRL (2009) - Flach, Chem. Phys (2010)]

Weak chaos: $m_2 \propto t^{1/5}$ **Strong chaos:** $m_2 \propto t^{1/3}$

Spreading in 2D DKG and 2D DDNLS

Excitation of central L×L sites and average over 50 realizations
[Many Manda, Senyange & S., PRE (2020)]

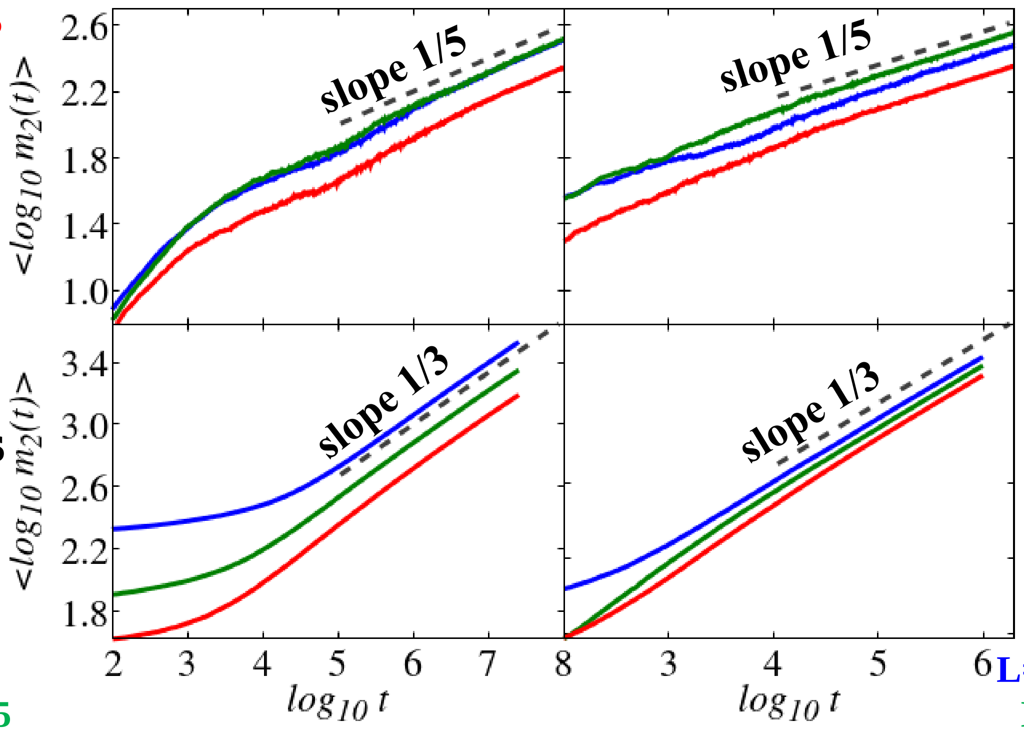
2D DKG | **2D DDNLS**

L=3, W=10, $E_{l,m}$=0.0085
L=1, W=10, $E_{l,m}$=0.05
L=2, W=11, $E_{l,m}$=0.0175

L=2, W=10, β=0.15, $s_{l,m}$=1
L=1, W=10, β=0.92, $s_{l,m}$=1
L=1, W=12, β=1.75, $s_{l,m}$=1

Weak chaos

Strong chaos



$$m_2 \propto t^{a_m}$$

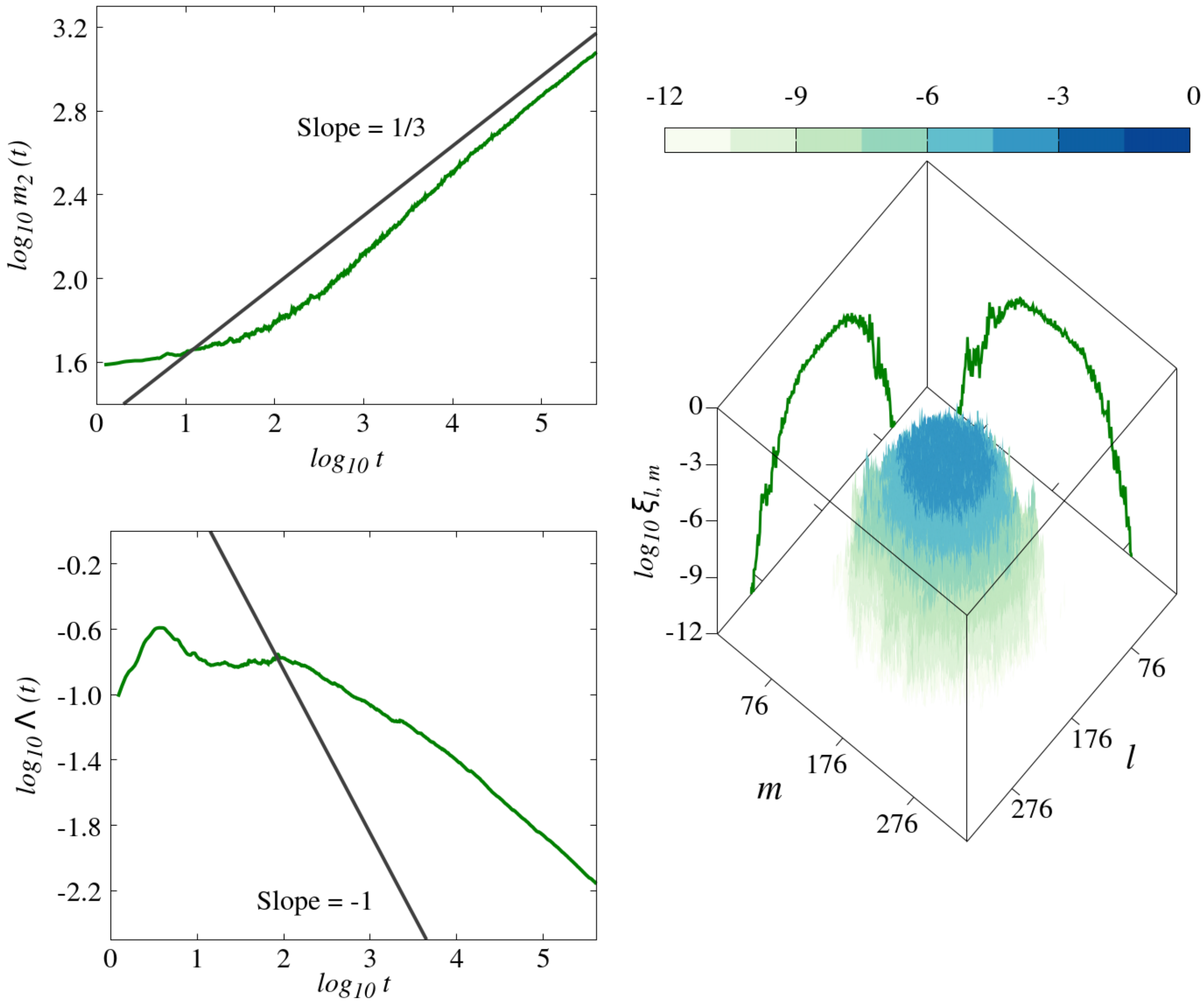
L=35, W=9, $E_{l,m}$=0.006
L=21, W=10, $E_{l,m}$=0.0135
L=15, W=12.5, $E_{l,m}$=0.035

L=21, W=10.5, β=0.145, $s_{l,m}$=1
L=10, W=11, β=0.68, $s_{l,m}$=1
L=15, W=14, β=6, $s_{l,m}$=1

The weak chaos case of the 2D DKG system was also considered in Laptyeva et al., EPL (2012)

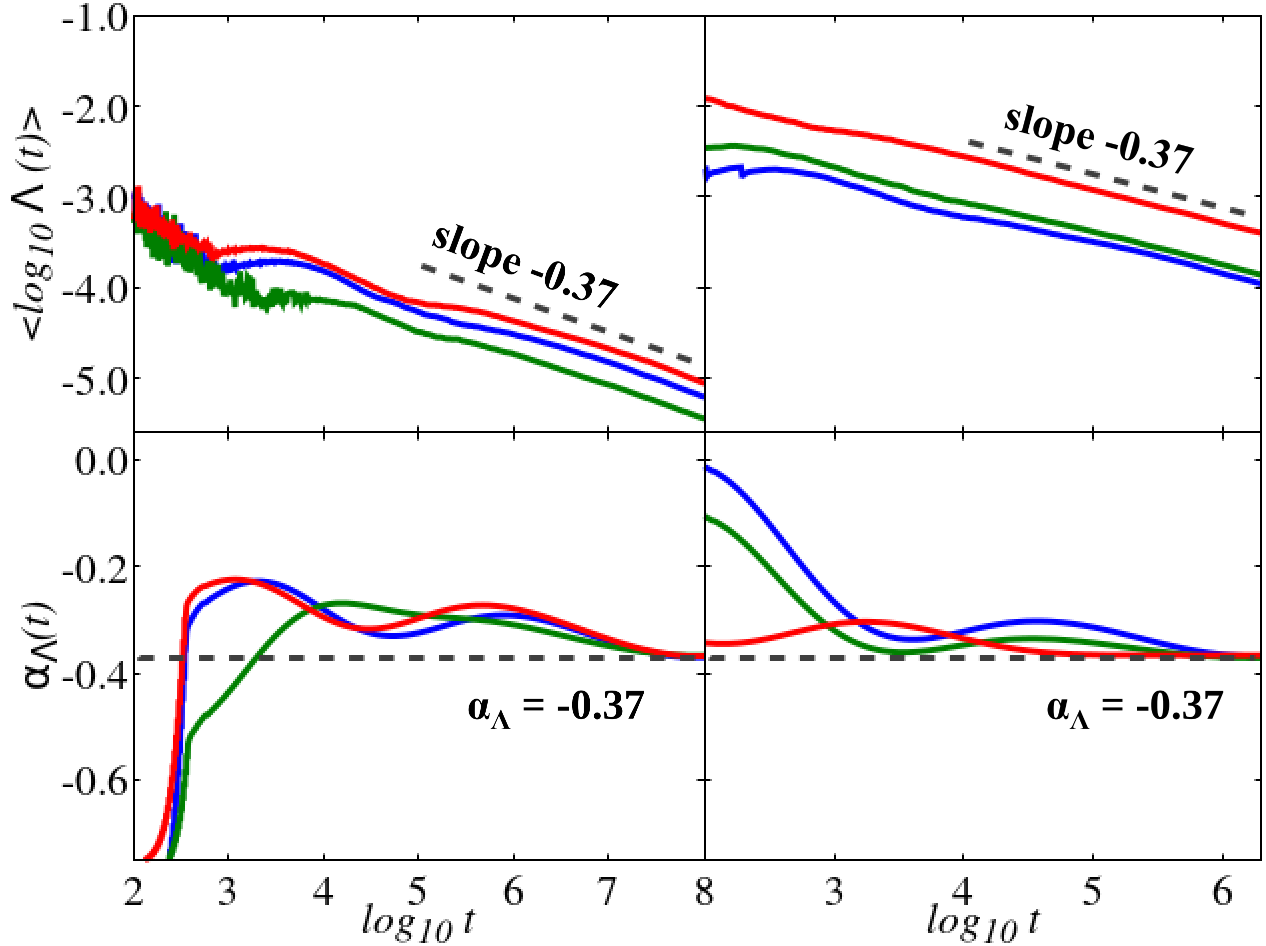
2D DDNLS : Strong Chaos

L=15, W=12, β=0.425, $s_{l,m}$=1, H_{2D}=1.32



Weak Chaos: 2D DKG and 2D DDNLS

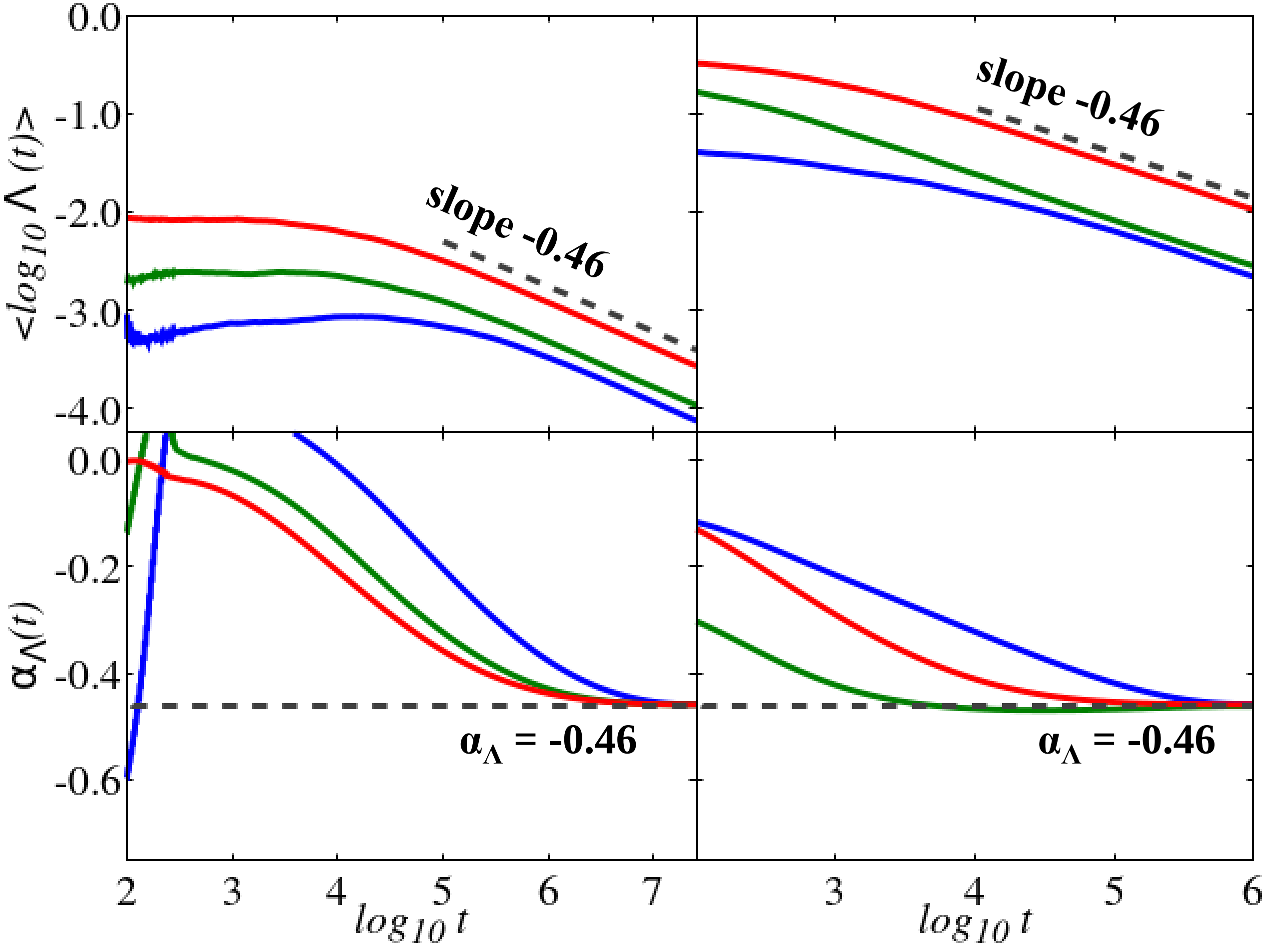
2D DKG   2D DDNLS



$\Lambda \propto t^{\alpha_\Lambda}$

Strong Chaos: 2D DKG and 2D DDNLS

2D DKG 2D DDNLS



$\Lambda \propto t^{\alpha_\Lambda}$

Dimension-independent scaling between chaoticity and spreading

Second moment: Theoretical predictions verified by numerical computations

$$m_2 \propto t^{a_m}$$

a_m	Weak	Strong
1D	1/3	1/2
2D	1/5	1/3

α_Λ	Weak	Strong
1D	-0.25	-0.30
2D	-0.37	-0.46

$$\Lambda \propto t^{\alpha_\Lambda}$$

Finite time mLCE: Numerical computations

For 1D and 2D systems there exists a uniform *scaling between the wave packet's spreading and its degree of chaoticity* indicating that nonlinear interactions of the same nature are responsible for the chaotic wave-packet spreading in both cases.

$$\left.\frac{\Lambda(t)}{m_2(t)}\right|_{1D} = \left.\frac{\Lambda(t)}{m_2(t)}\right|_{2D}$$

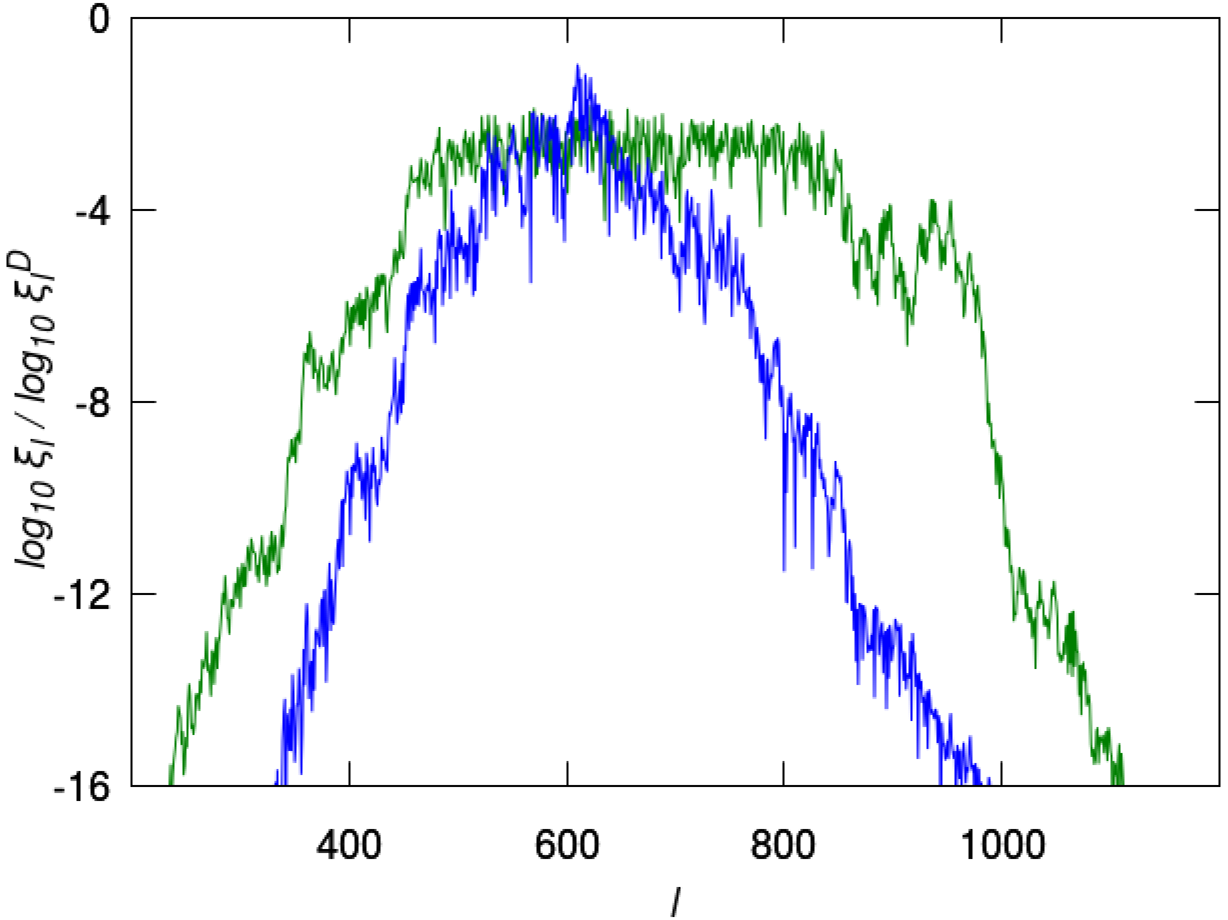
Weak chaos

$$t^{-0.58} \approx t^{-0.57}$$

Strong chaos

$$t^{-0.80} \approx t^{-0.79}$$

1D: Deviation Vector Distributions (DVDs)



Energy
DVD

1D DKG
weak chaos
L=37 sites,
H_{1K}=0.37,
W=3

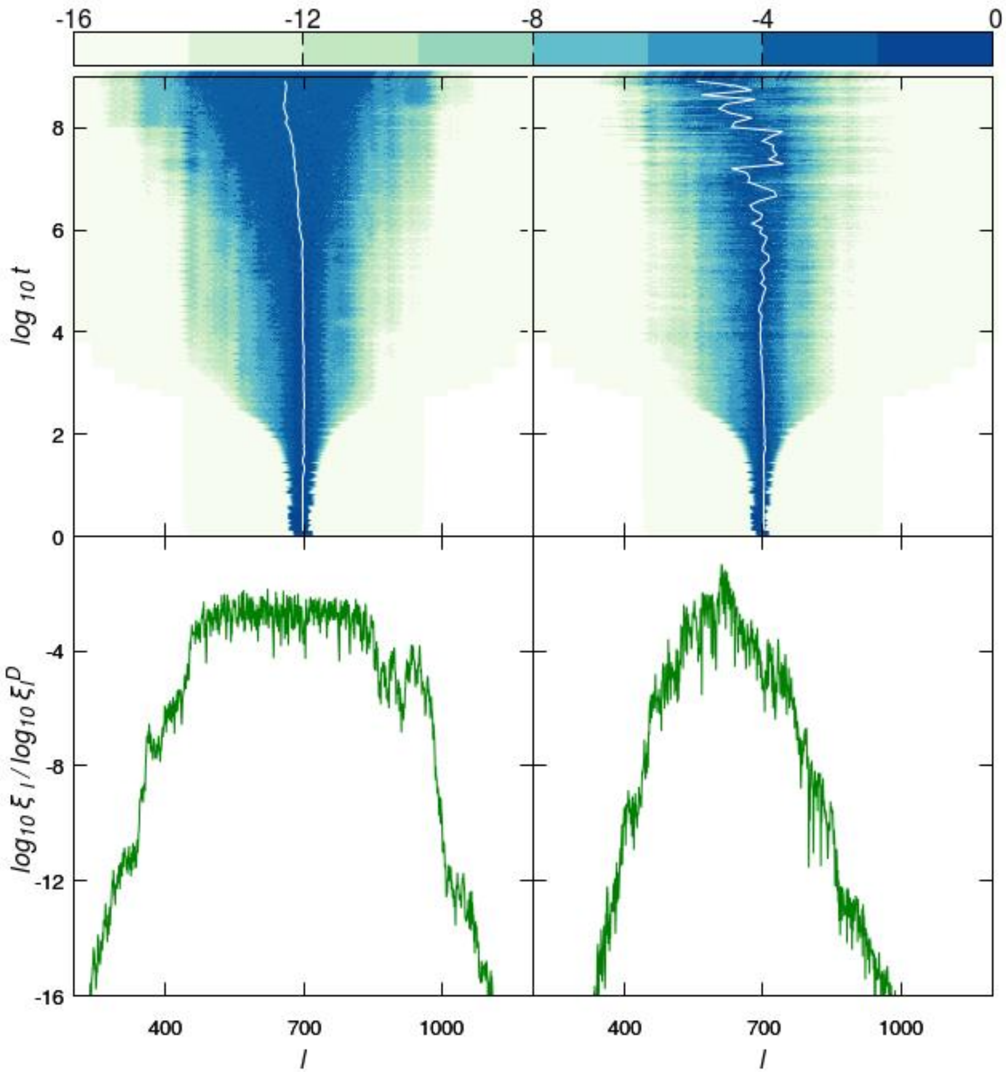
Deviation vector:
$v(t)=(\delta u_1(t), \delta u_2(t),\ldots, \delta u_N(t), \delta p_1(t), \delta p_2(t),\ldots, \delta p_N(t))$

DVD: $$\xi_l^D = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$$

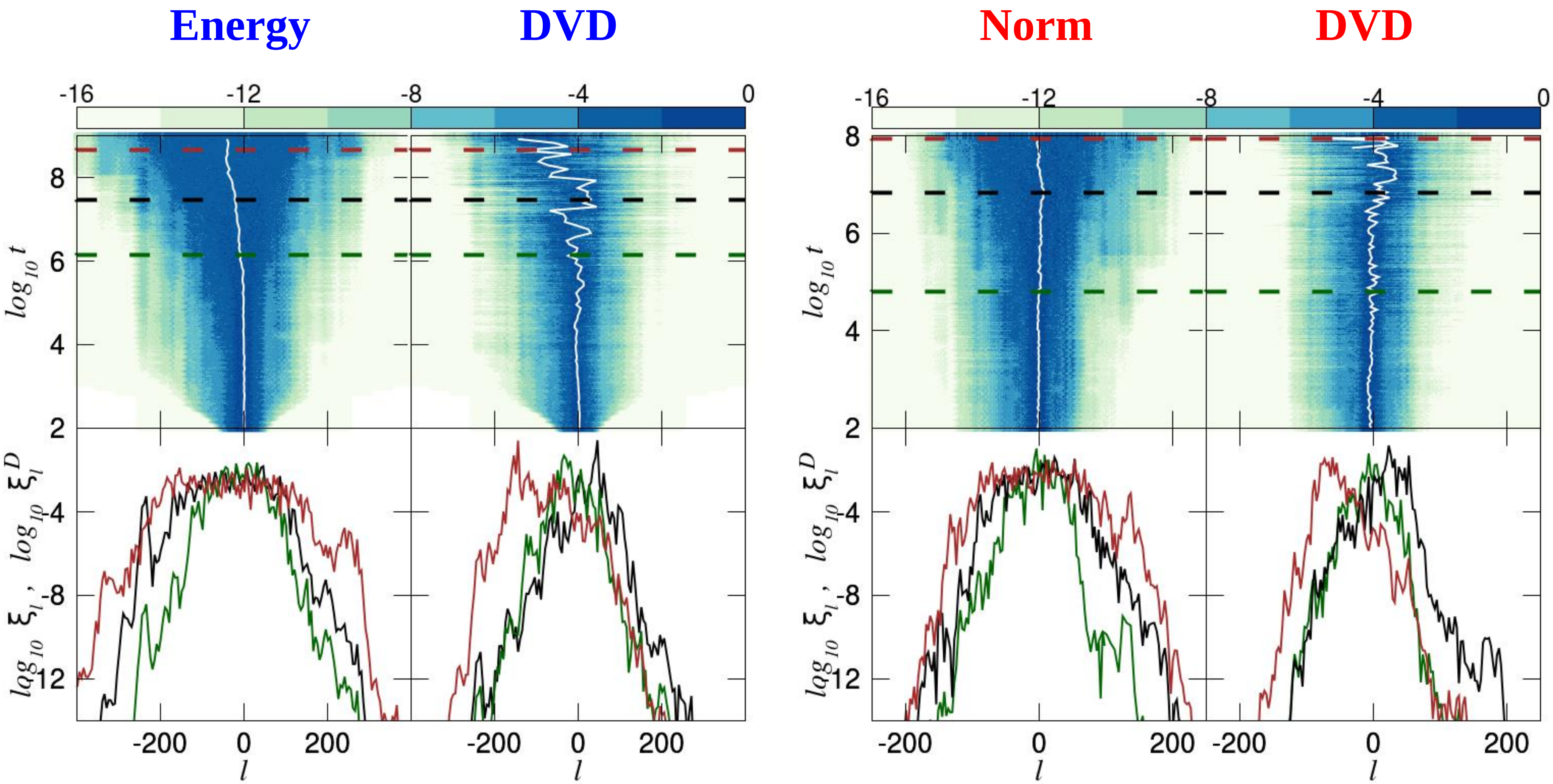
1D: Deviation Vector Distributions (DVDs)

1D DKG: weak chaos. L=37 sites, H_{1K}=0.37, W=3

Energy **DVD**



Weak Chaos (1D): DKG and DDNLS



DKG: W=3, L=37, H_{1K}=0.37

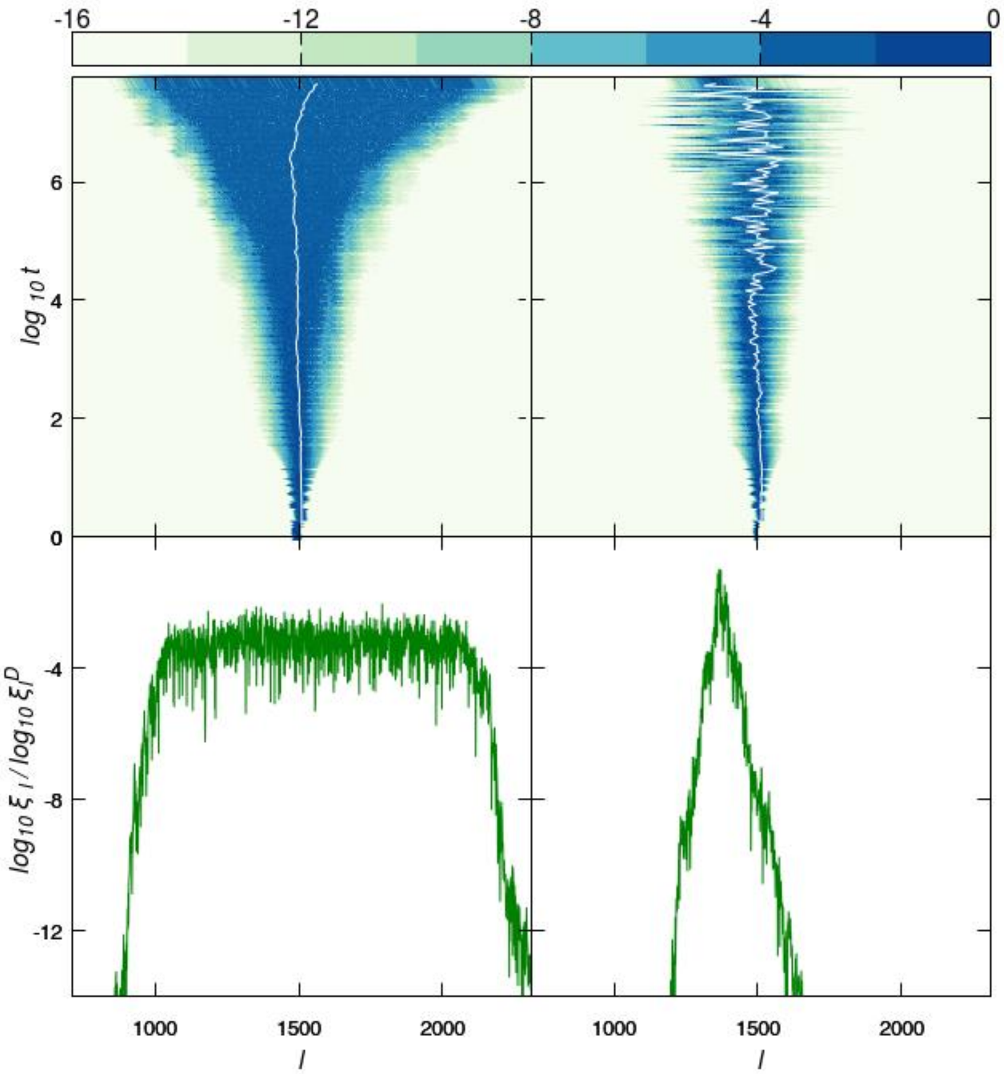
DDNLS: W=4, L=21, β=0.04

1D: Deviation Vector Distributions (DVDs)

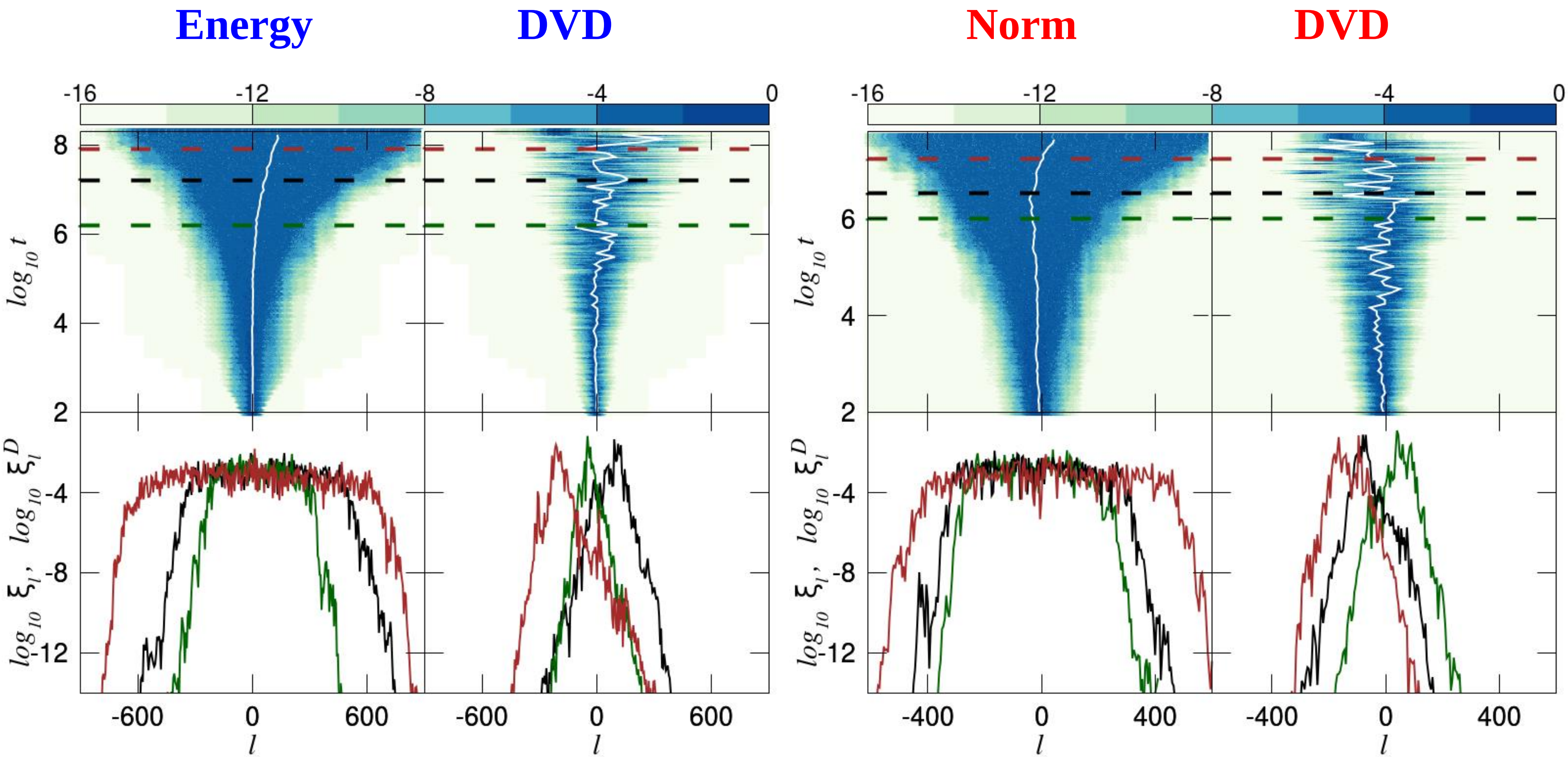
1D DDNLS: strong chaos W=3.5, L=21, β=0.72

Norm

DVD



Strong Chaos (1D): DKG and DDNLS



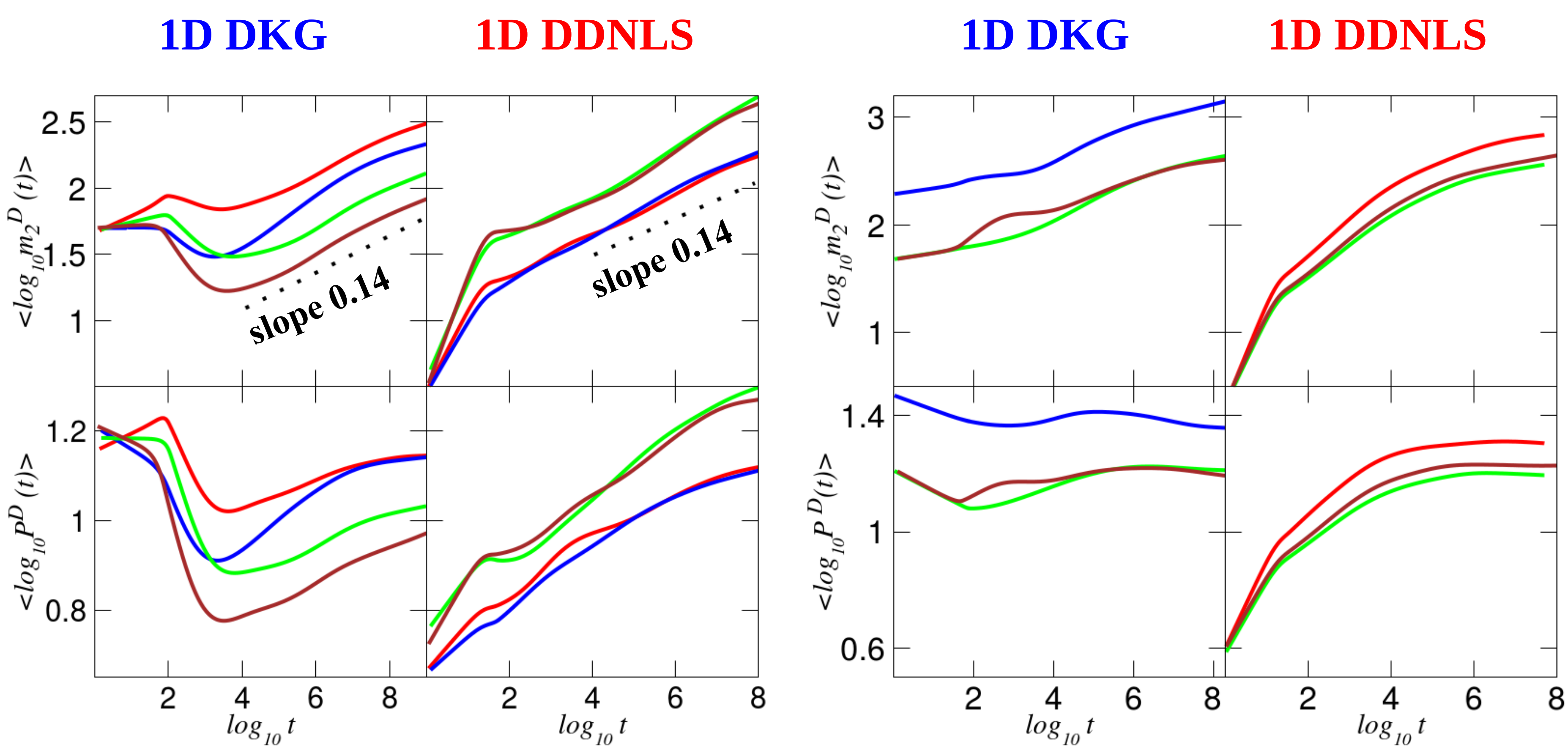
DKG: W=3, L=83, H_{1K}=8.3

DDNLS: W=3.5, L=21, β=0.72

1D: Characteristics of DVDs

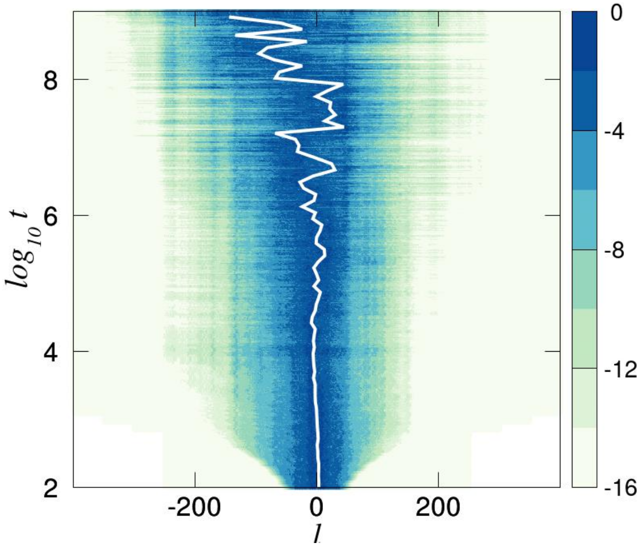
Weak chaos

Strong chaos



1D: Characteristics of DVDs

1D DKG weak chaos
L=37, H_{1K}=0.37, W=3

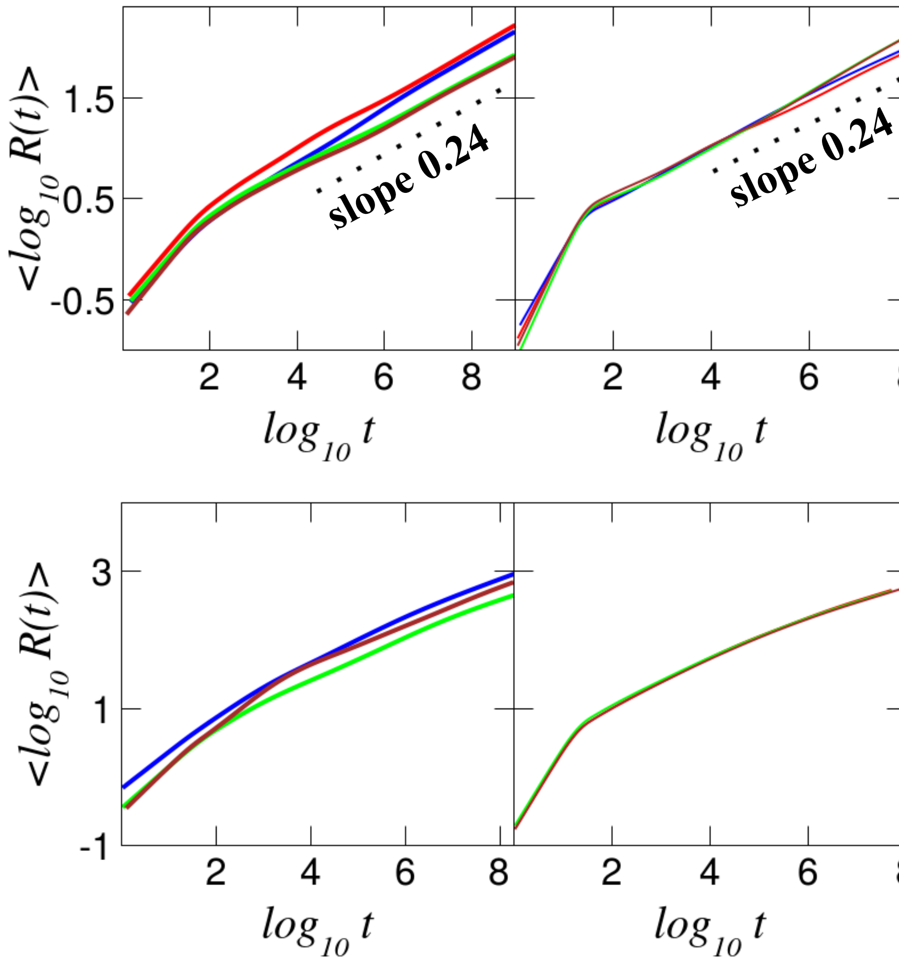


Range of the lattice visited by the DVD

$$R(t) = \max_{[0,t]}\left\{\bar{l}_w(t)\right\} - \min_{[0,t]}\left\{\bar{l}_w(t)\right\}$$

$$\bar{l}_w = \sum_{l=1}^{N} l \xi_l^D$$

1D DKG **1D DDNLS**



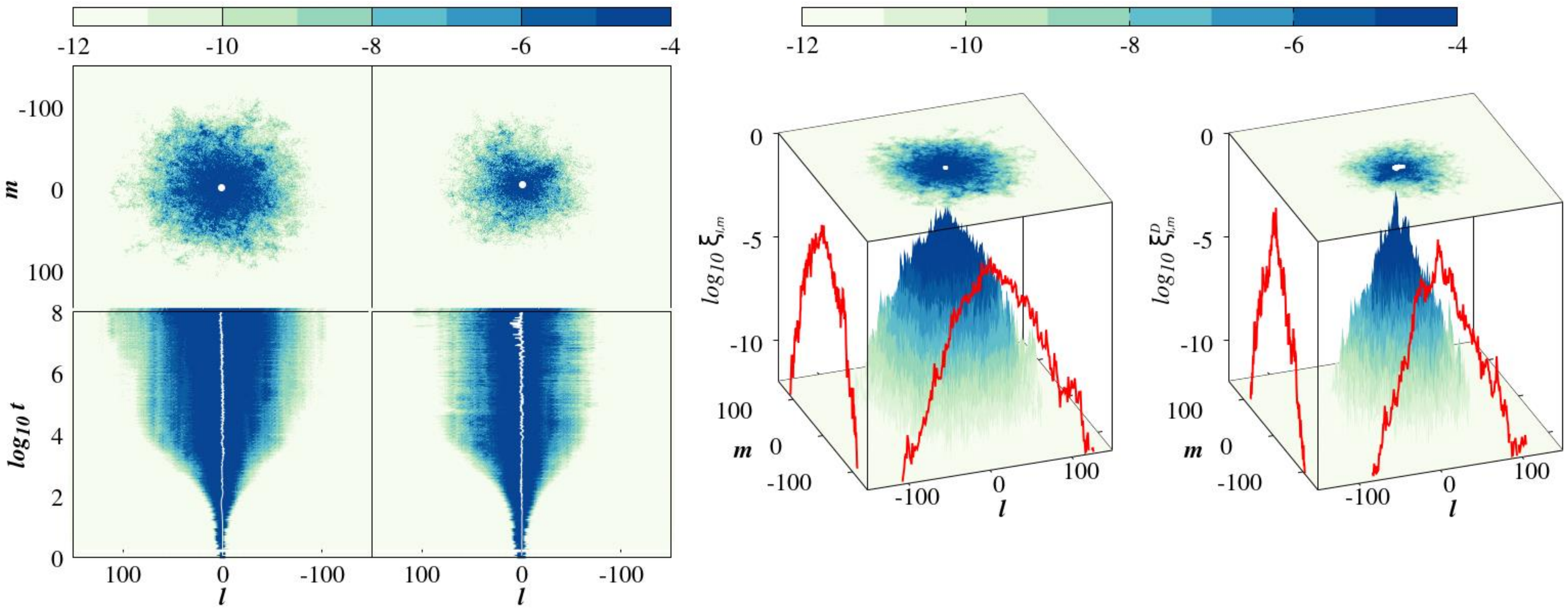
Weak chaos

Strong chaos

2D: Deviation Vector Distributions (DVDs)

2D DKG: weak chaos

L=1 sites, H_{2K}=0.05, W=10

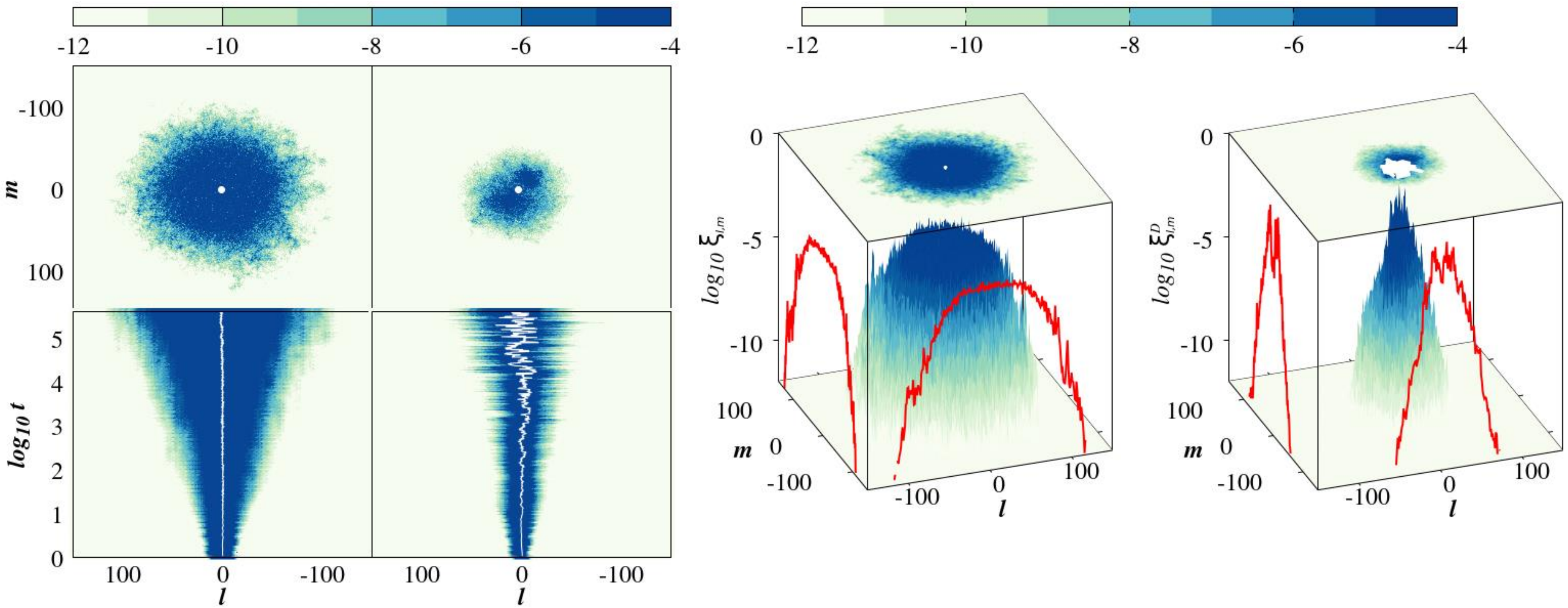


Energy **DVD** **Energy** **DVD**

2D: Deviation Vector Distributions (DVDs)

2D DDNLS: strong chaos

L=15, W=12, β=0.425, $s_{l,m}$=1, H_{2D}=1.32



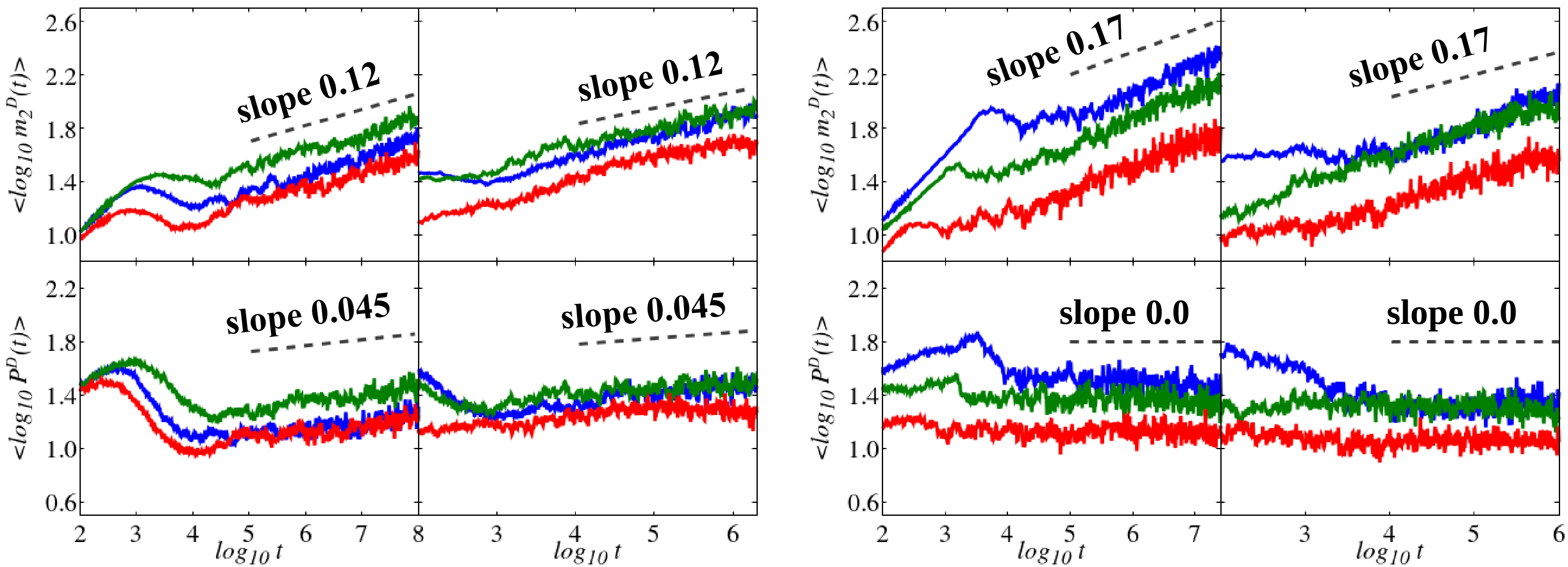
Norm **DVD** **Norm** **DVD**

2D: Characteristics of DVDs

Weak chaos

Strong chaos

2D DKG | 2D DDNLS | 2D DKG | 2D DDNLS

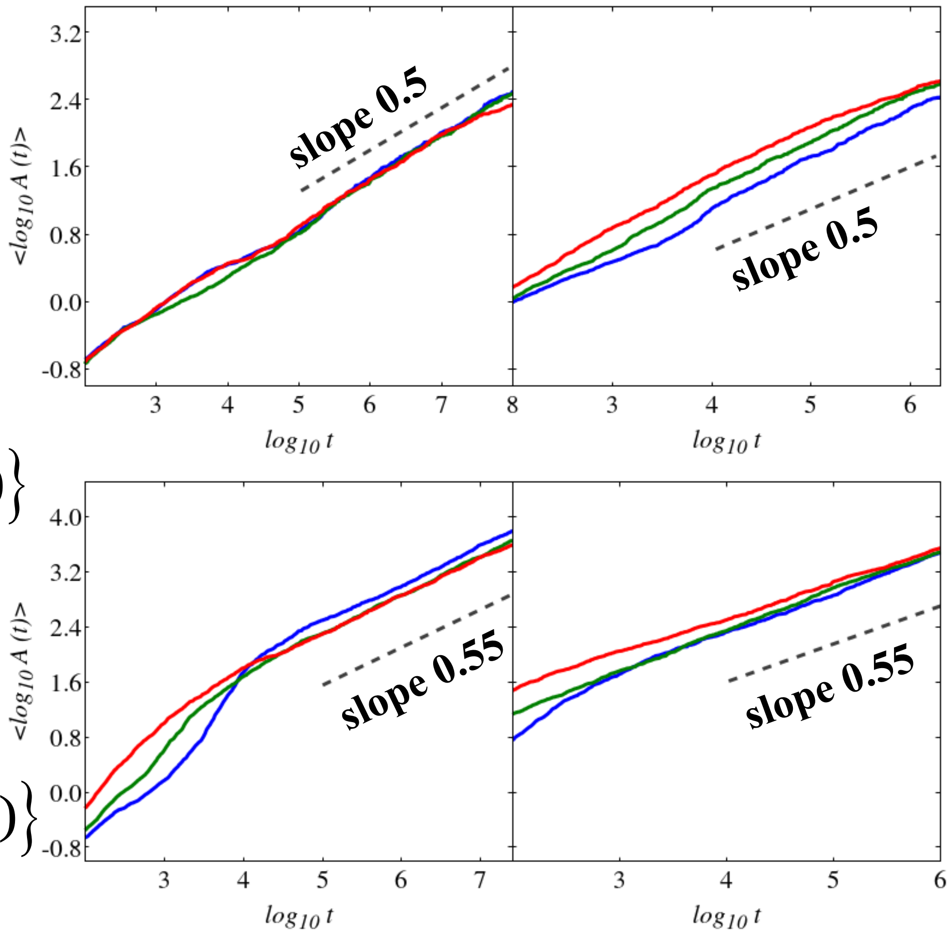


2D: Characteristics of DVDs

2D DKG **2D DDNLS**

Area of the lattice visited by the DVD

$$A(t)=R_x(t)\cdot R_y(t)$$



Weak chaos

Strong chaos

$$R_x(t)=\max_{[0,t]}\left\{\overline{l}^{\,D}(t)\right\}-\min_{[0,t]}\left\{\overline{l}^{\,D}(t)\right\}$$

$$\overline{l}^{\,D}=\sum_{l,m} l\,\xi^{D}_{l,m}$$

$$R_y(t)=\max_{[0,t]}\left\{\overline{m}^{D}(t)\right\}-\min_{[0,t]}\left\{\overline{m}^{D}(t)\right\}$$

$$\overline{m}^{D}=\sum_{l,m} m\,\xi^{D}_{l,m}$$

Summary

We investigated in depth the chaotic wave-packet spreading in 1D and 2D disordered nonlinear systems

- **We verified theoretical predictions for the characteristics of spreading in 2D**
  - **strong chaos regime, weak chaos for the DDNLS system**
- **Generality of results for 1D and 2D systems**
  - **both the DKG and the DDNLS models show similar chaotic behaviors for each dynamical regime (weak – strong chaos)**
- **Universal decrease of the systems' chaoticity in time**
  - **1D: Weak chaos: $\Lambda \propto t^{-0.25}$ - Strong chaos: $\Lambda \propto t^{-0.30}$**
  - **2D: Weak chaos: $\Lambda \propto t^{-0.37}$ - Strong chaos: $\Lambda \propto t^{-0.46}$**
- **Dimension-independent scaling between the wave packet's spreading and chaoticity: Λ/m_2 (1D) = Λ/m_2 (2D). What about 3D?**
- **The DVDs provide information about the propagation of chaos**
  - **wandering of localized chaotic hot spots in the lattice's excited part homogenizes chaos**

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