Matrix Product States and Exact Solutions of One dimensional Stochastic Processes

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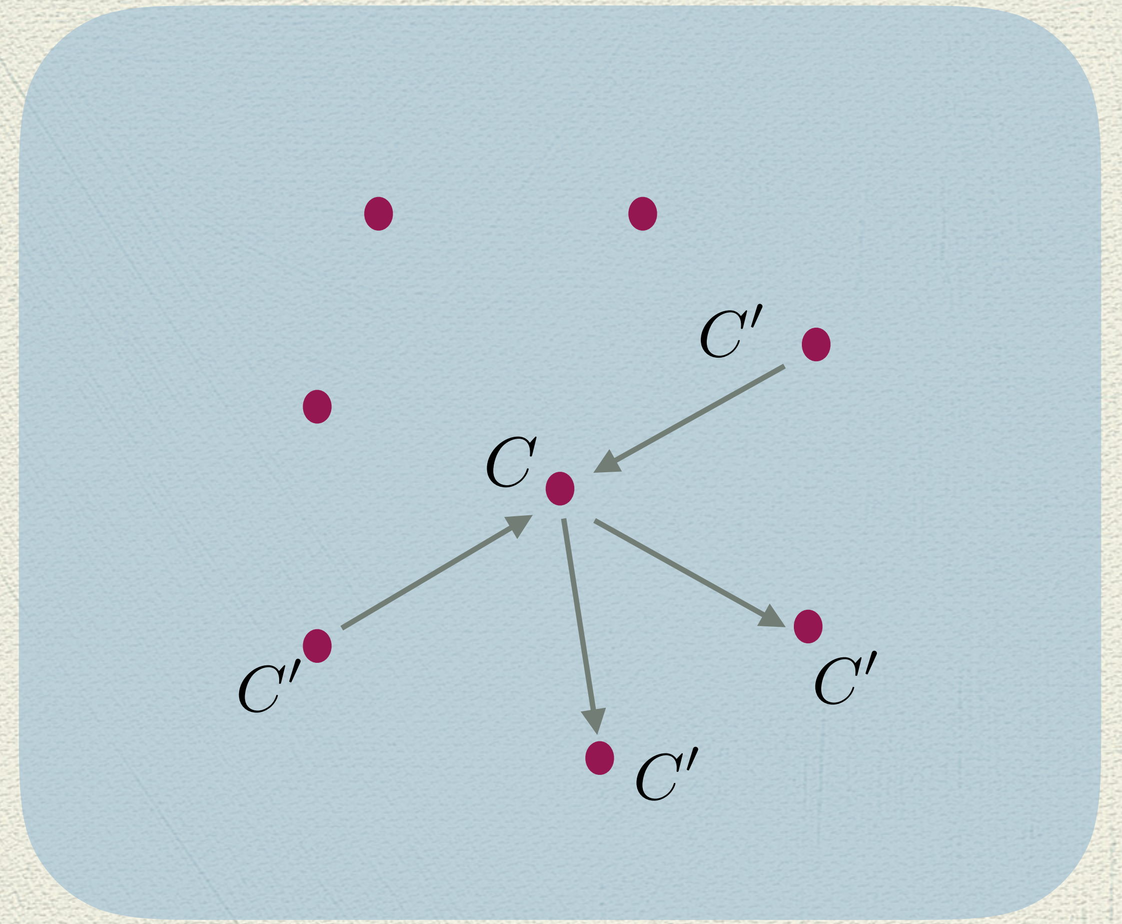
Hamiltonian Formulation of Stochastic Processes

$$\frac{\partial P(C)}{\partial t} = \sum_{C'} P(C', t) w(C' \to C) - \sum_{C'} P(C) w(C \to C')$$

$$\frac{\partial}{\partial t} |P\rangle = -H|P\rangle$$

$$H|P\rangle = 0$$



Hamiltonian Formulation of Stochastic Processes

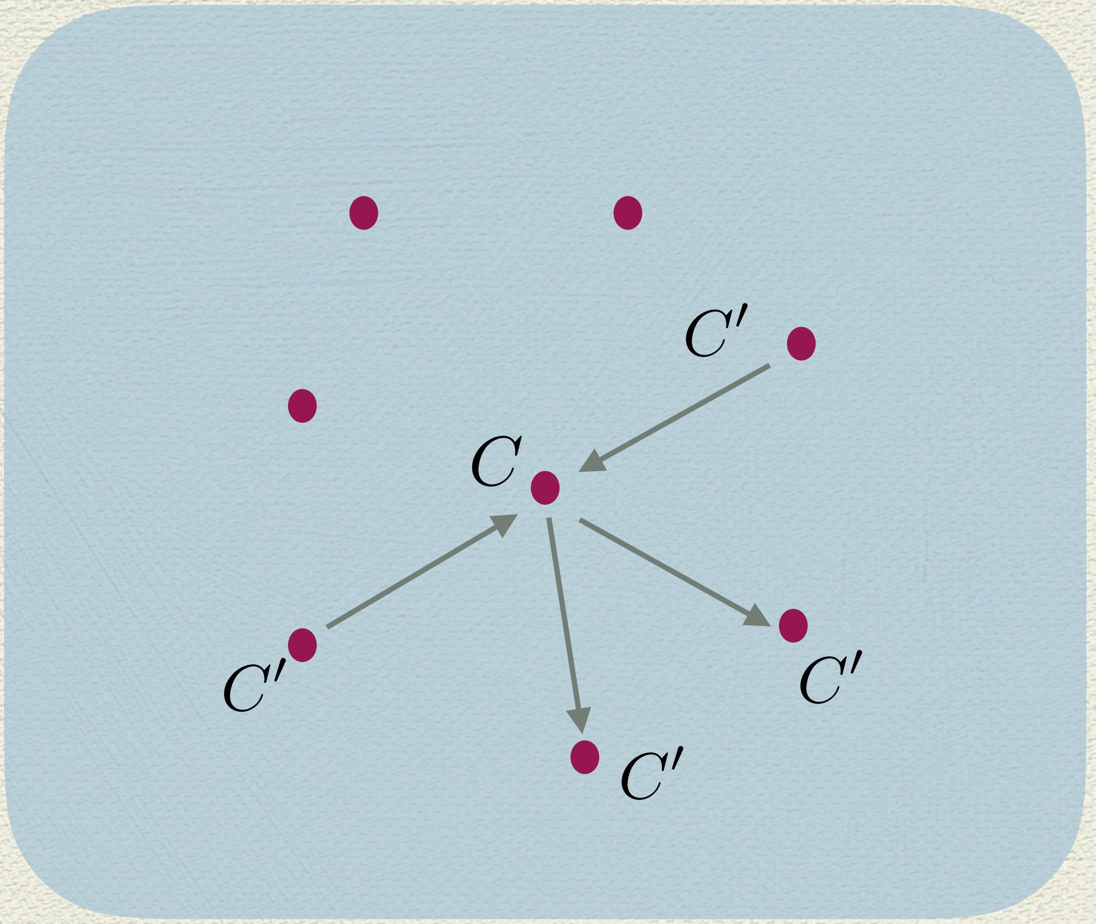
$$\frac{\partial}{\partial t}|P\rangle = -H|P\rangle$$

$$w(C \to C') \geq 0$$

$$\sum_{C'} w(C \to C') = 0$$

$$\lambda_0 = 1$$

$$Re(\lambda_i) > 0$$



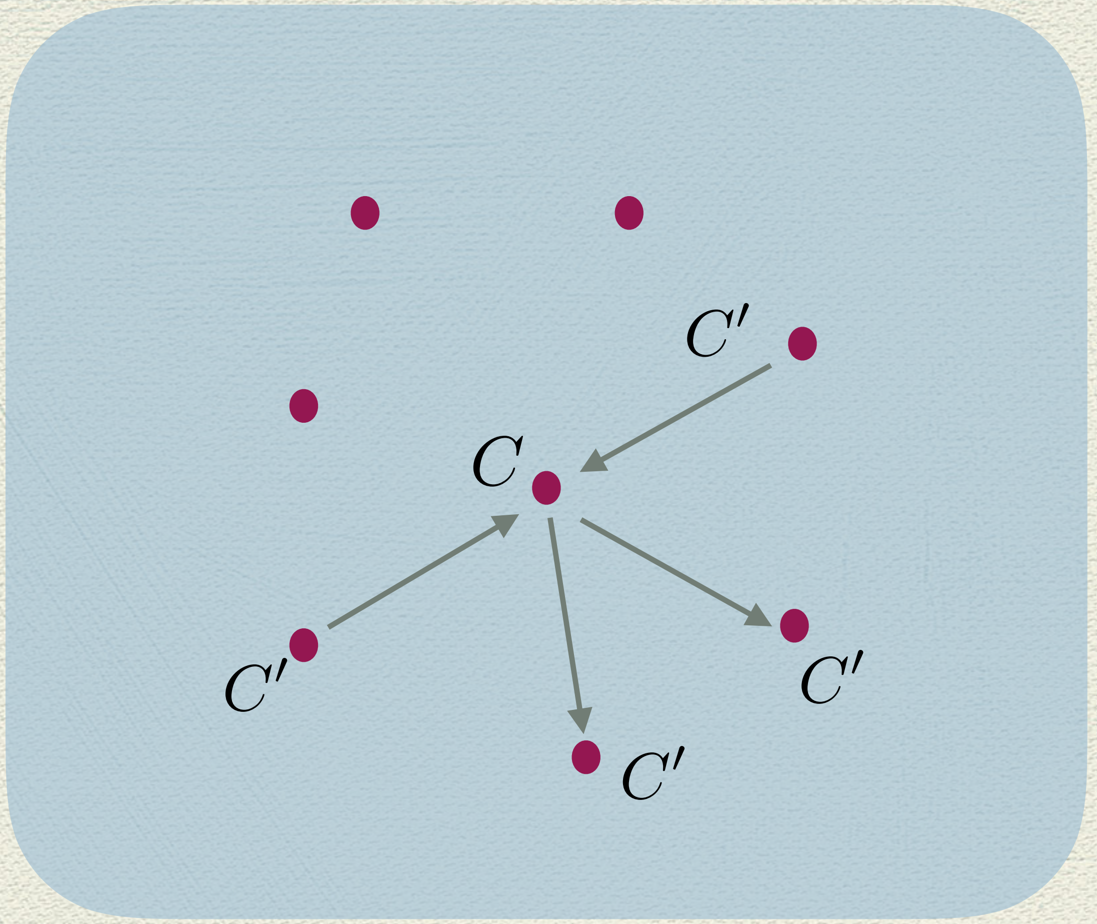
Hamiltonian Formulation of Stochastic Processes

$$\lambda_0 = 1$$

$$Re(\lambda_i) > 0$$

$$|P(t)\rangle = e^{-tH}|P(0)\rangle$$

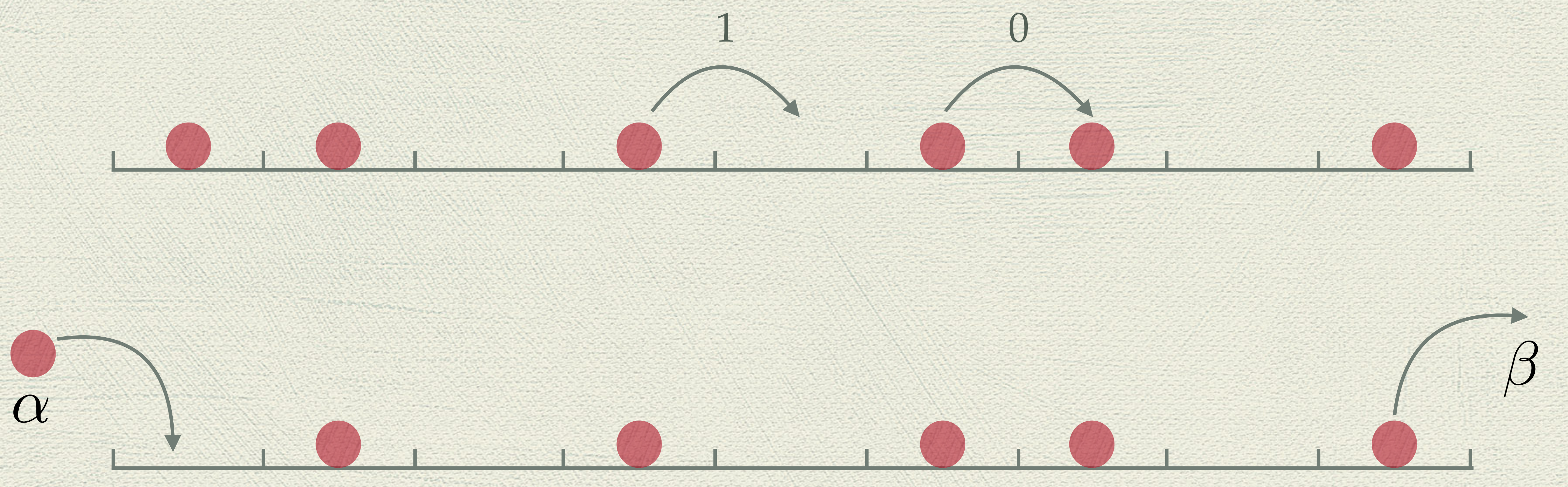
$$|P(\infty)\rangle = |\Psi_0\rangle$$



$$H|\Psi_0\rangle = 0$$

Steady State

Asymmetric Simple Exclusion Process (ASEP)

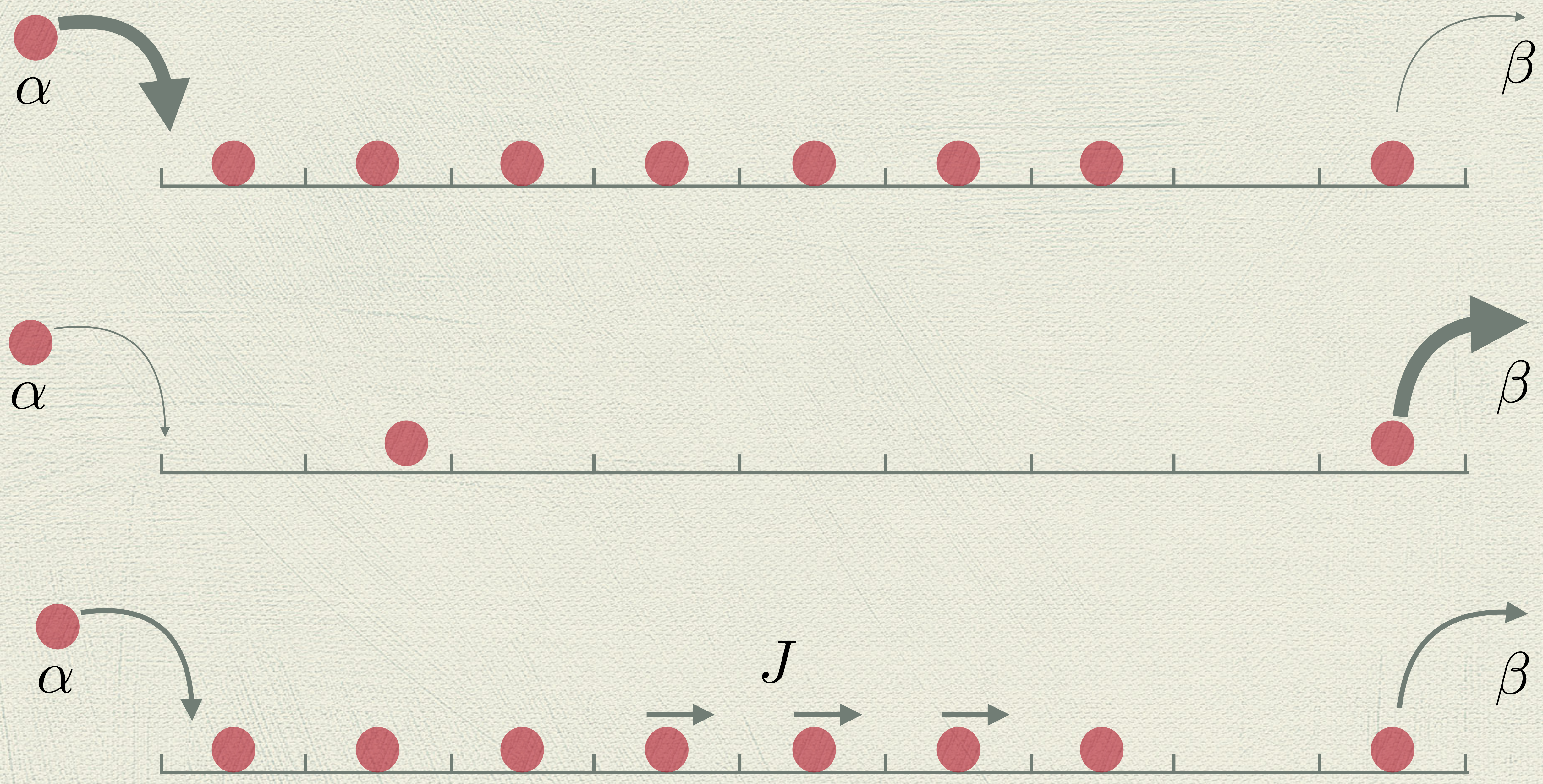


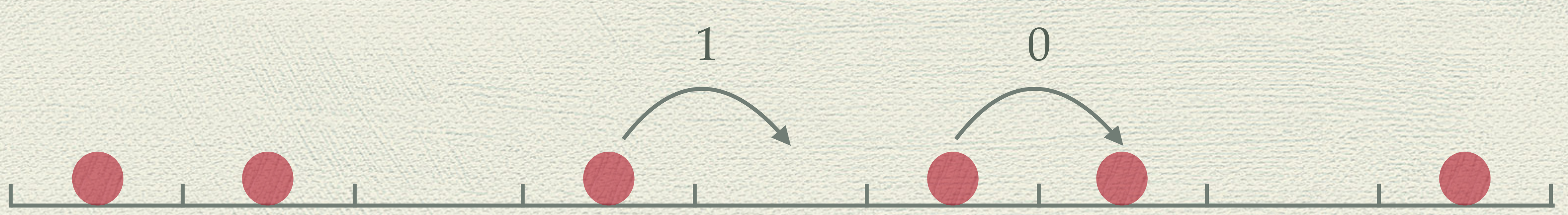
On the average from n particles at a given site,

$$n \times dt$$

particles hop to the empty site ahead.

Asymmetric Simple Exclusion Process (ASEP)





Configurations $\longrightarrow$ $C = (n_1, n_2, \cdots n_L)$ $\qquad n_i = 0\ ,\ 1$

$$P(n_1, n_2, \cdots n_L; t)$$

$$H_{bulk} = \sum_{i=1}^{N-1} -\sigma_i^- \sigma_{i+1}^+ + \hat{n}_i(1 - \hat{n}_{i+1}).$$

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$$h_L := -\alpha(\sigma_1^+ - (1 - \hat{n}_1))$$

$$h_R := -\beta(\sigma_N^- - \hat{n}_N)$$

This Hamiltonian is not Hermitian.

The Basic Problem

$$H|\Psi_0\rangle = 0$$

↑

Find the Steady State

The Matrix Product State

$$P(n_1, n_2, \cdots n_L) = \langle W|A_{n_1} A_{n_2} \cdots A_{n_L}|V\rangle$$

$$P(n_1, n_2, \cdots n_L) = \frac{1}{Z_L} \langle W | A_{n_1} A_{n_2} \cdots A_{n_L} | V \rangle$$

○ E The matrix for filled site

● D The matrix for empty site

$$P(\bullet\bullet\circ\bullet\bullet\circ\circ\cdots) = \frac{1}{Z_L} \langle W | D^2 E D^2 E^2 \cdots | V \rangle$$

$$Z_L = \langle W | (D + E)^L | V \rangle$$

$$Z_L = \langle W|(D+E)^L|V\rangle$$

Partition Function = A sum over all configurations

Partition Function → All steady state properties

How to find the matrices E and D and the vectors

$$|V\rangle \quad \text{and} \quad \langle W|$$

The Physical Process → **The Algebra**

$$DE = D + E$$

$$D|V\rangle = \frac{1}{\beta}|V\rangle$$

$$\langle W|E = \frac{1}{\alpha}\langle W|$$

$$Z_L = \langle W|(D+E)^L|V\rangle$$

$$D + E = \mathcal{C}$$

$$Z_L = \langle W|\mathcal{C}^L|V\rangle$$

$$\langle n_k\rangle = \frac{1}{Z}\sum_C P(\cdots,1,\cdots)$$

$$\langle n_k\rangle = \frac{1}{Z_L}\langle W|\mathcal{C}^{k-1}D\mathcal{C}^{L-k}|V\rangle$$

$$J = \langle n_K(1 - n_{k+1}) \rangle$$

$$J = \frac{1}{Z_L} \langle W | \mathcal{C}^{k-1} D E \mathcal{C}^{L-k-1} | V \rangle$$

$$J = \frac{1}{Z_L} \langle W | \mathcal{C}^{k-1} (D + E) \mathcal{C}^{L-k-1} | V \rangle$$

Steady State Current → $J = \frac{Z_{L-1}}{Z_L}$

Analysis of the algebra

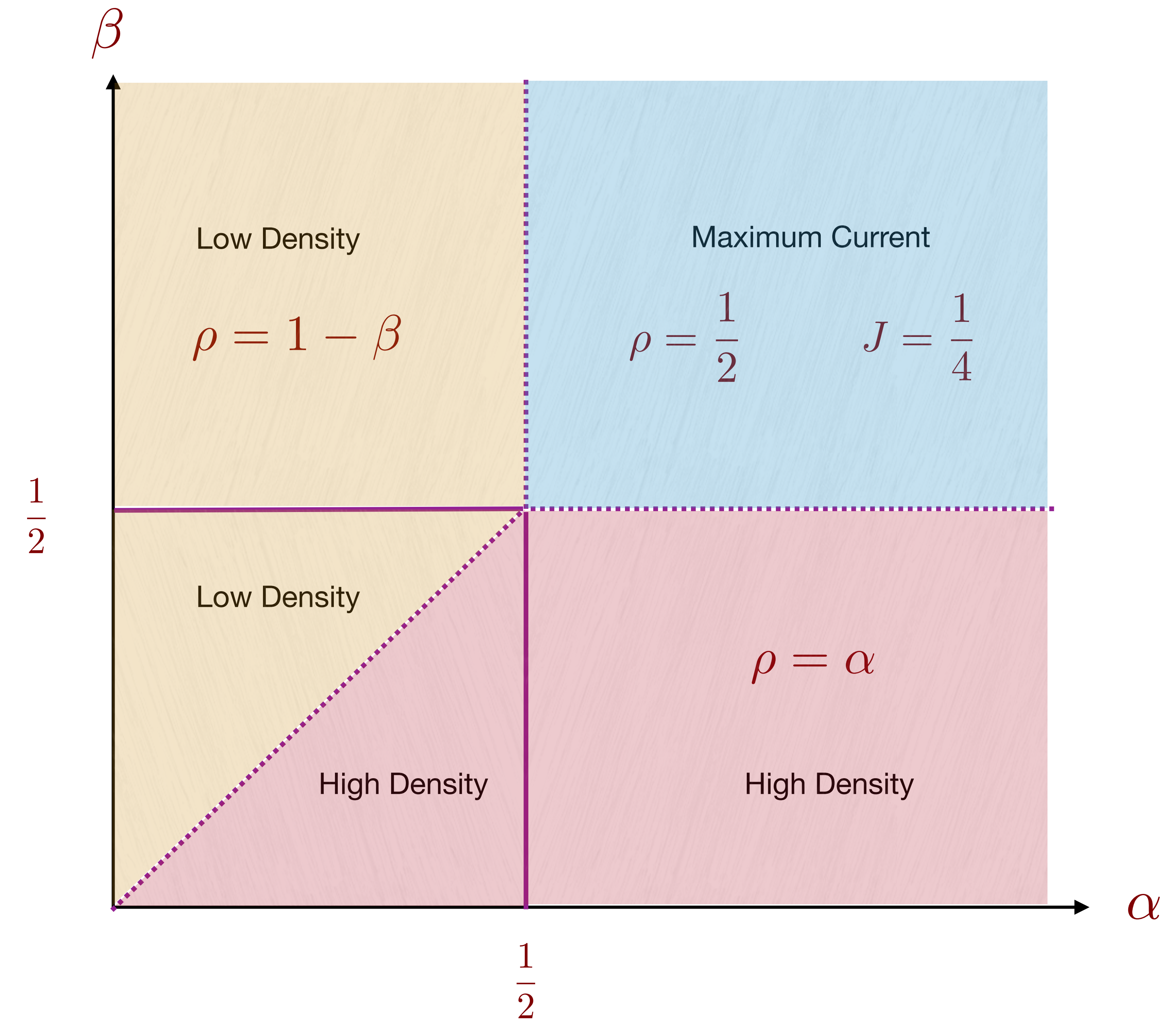
$$DE = D + E$$

1-D representation → **Mean Field**

$$D|V\rangle = \frac{1}{\beta}|V\rangle$$

$$\langle W|E = \frac{1}{\alpha}\langle W|$$

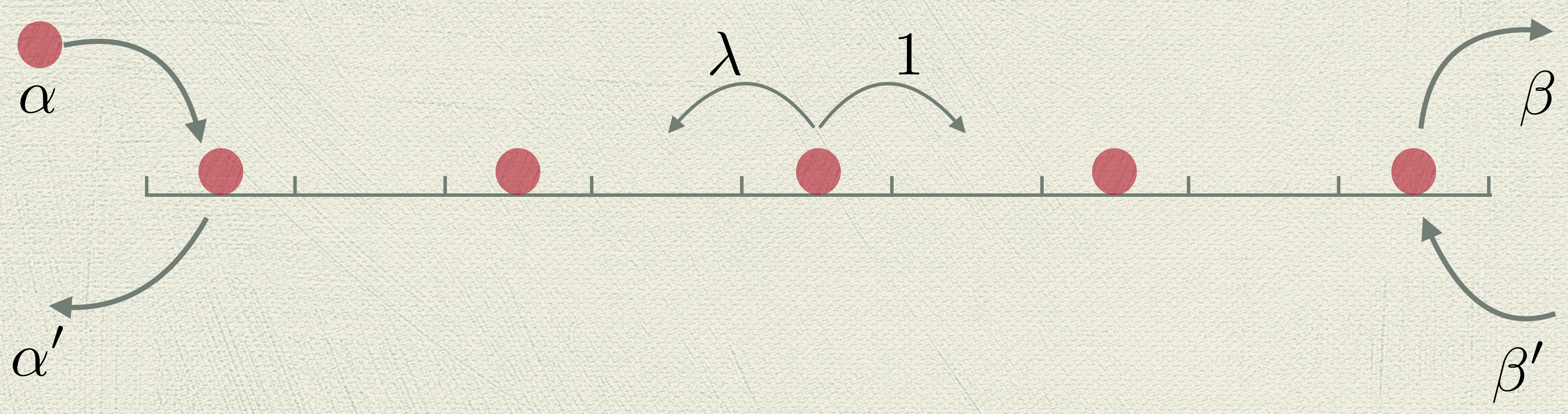
Infinite dimensional Represenation



B Derrida, M R Evans, V Hakim and V Pasquier, Journal of Physics A (1999).

Question:
Is it possible to generalize this process and its exact solution?

Making the process partially asymmetric?

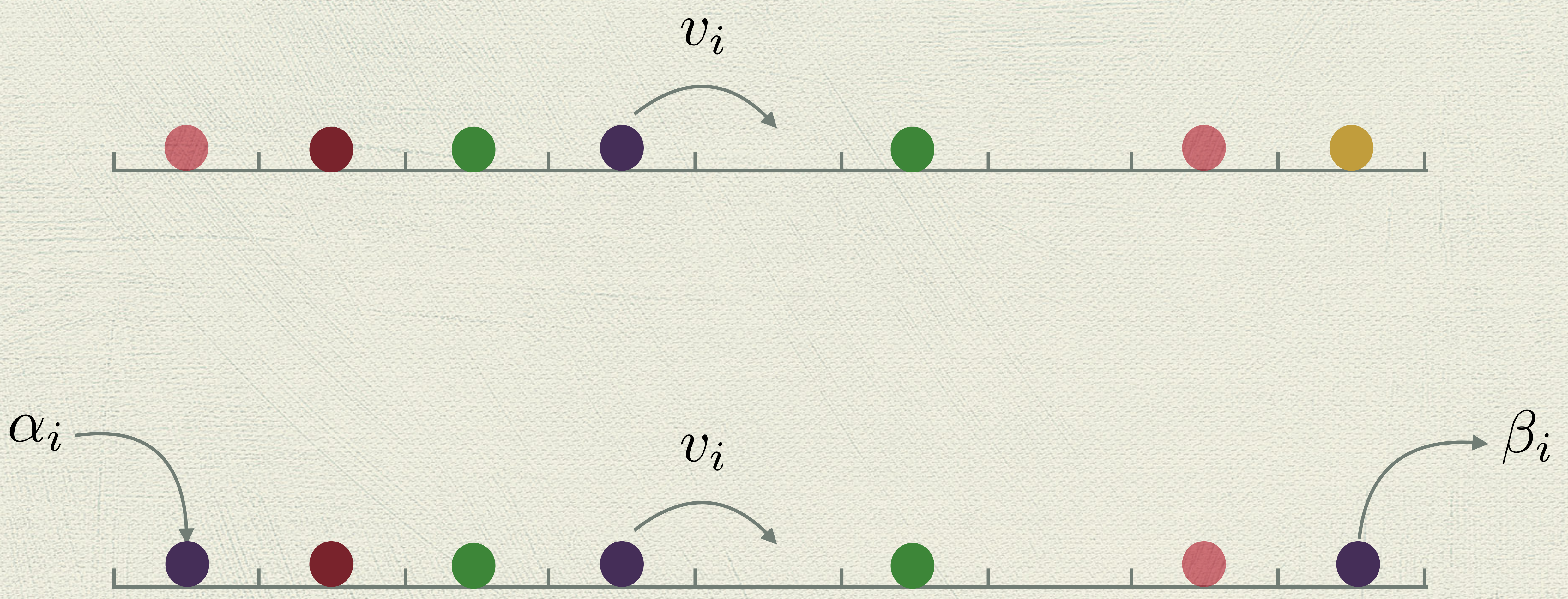


Adding other microscopic processes?

Pair annihilation

Pair Creation

Different types of particles



The basic Challenge

Defining a non-trivial process which is still Exactly Solvable.

How many particles?

The rates of hopping?

The rates of interaction?

The rates of injection and extraction?

The solution: Mathematical Beauty

Follow the reverse path and
Start from a consistent Algebra

- E
- D_1
- D_2
- D_3
- $\vdots$
- D_p

$$D_i E = ?$$

$$D_j D_i = ?$$

Demands:

$$D_i(D_j E) = (D_i D_j) E$$

$$D_i(D_j D_k) = (D_i D_j) D_k$$

The solution: Mathematical Beauty

There is only one consistent algebra!

- E
- D_1
- D_2
- D_3
- $\vdots$
- D_p

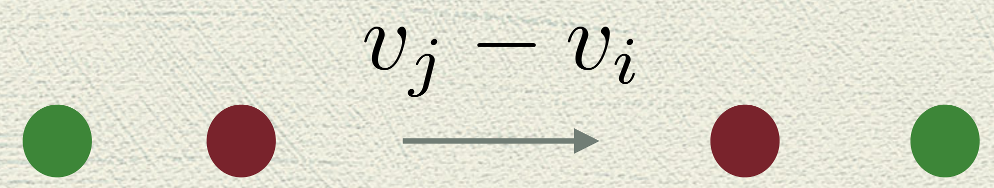
$$D_i E = \frac{1}{v_i} D_i + E$$

$$D_j D_i = \frac{v_i D_j - v_j D_i}{v_i - v_j} \qquad j > i$$

What are the meaning of these parameters?

Go backward and find the Hamiltonian

$$D_i E = \frac{1}{v_i} D_i + E$$

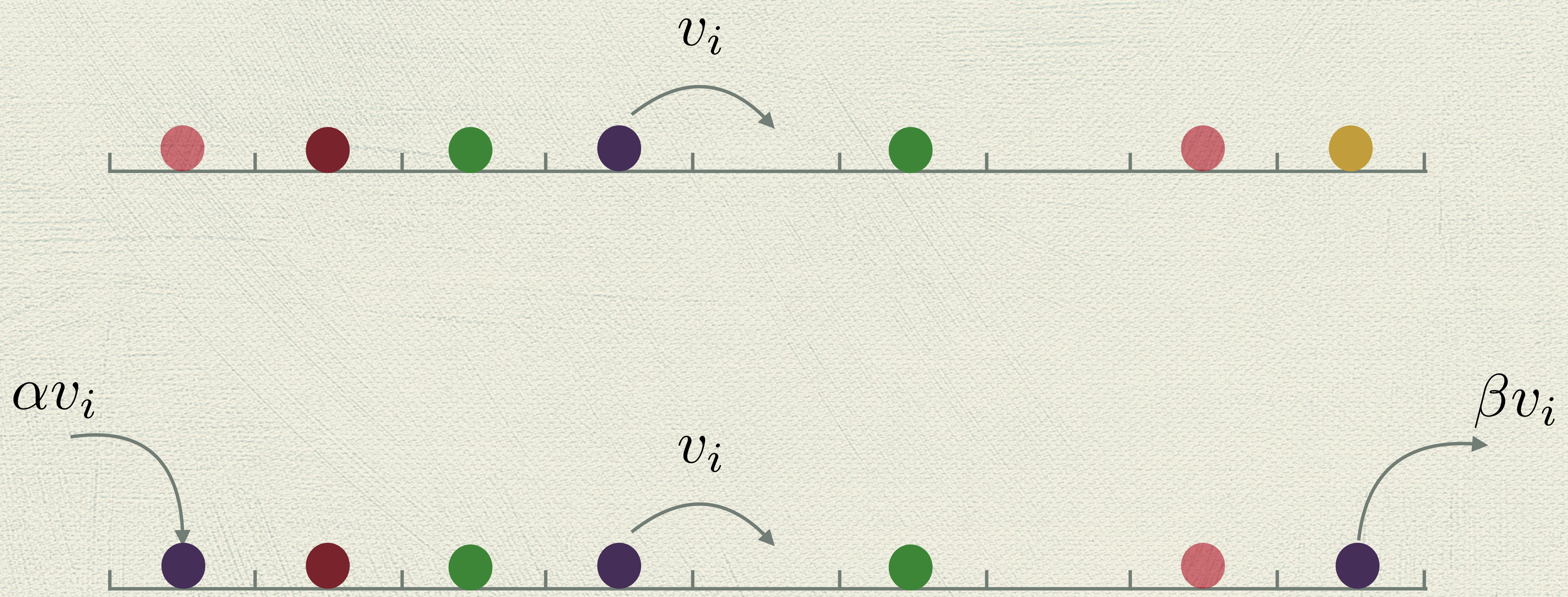


$$D_j D_i = \frac{v_i D_j - v_j D_i}{v_i - v_j} \qquad j > i$$

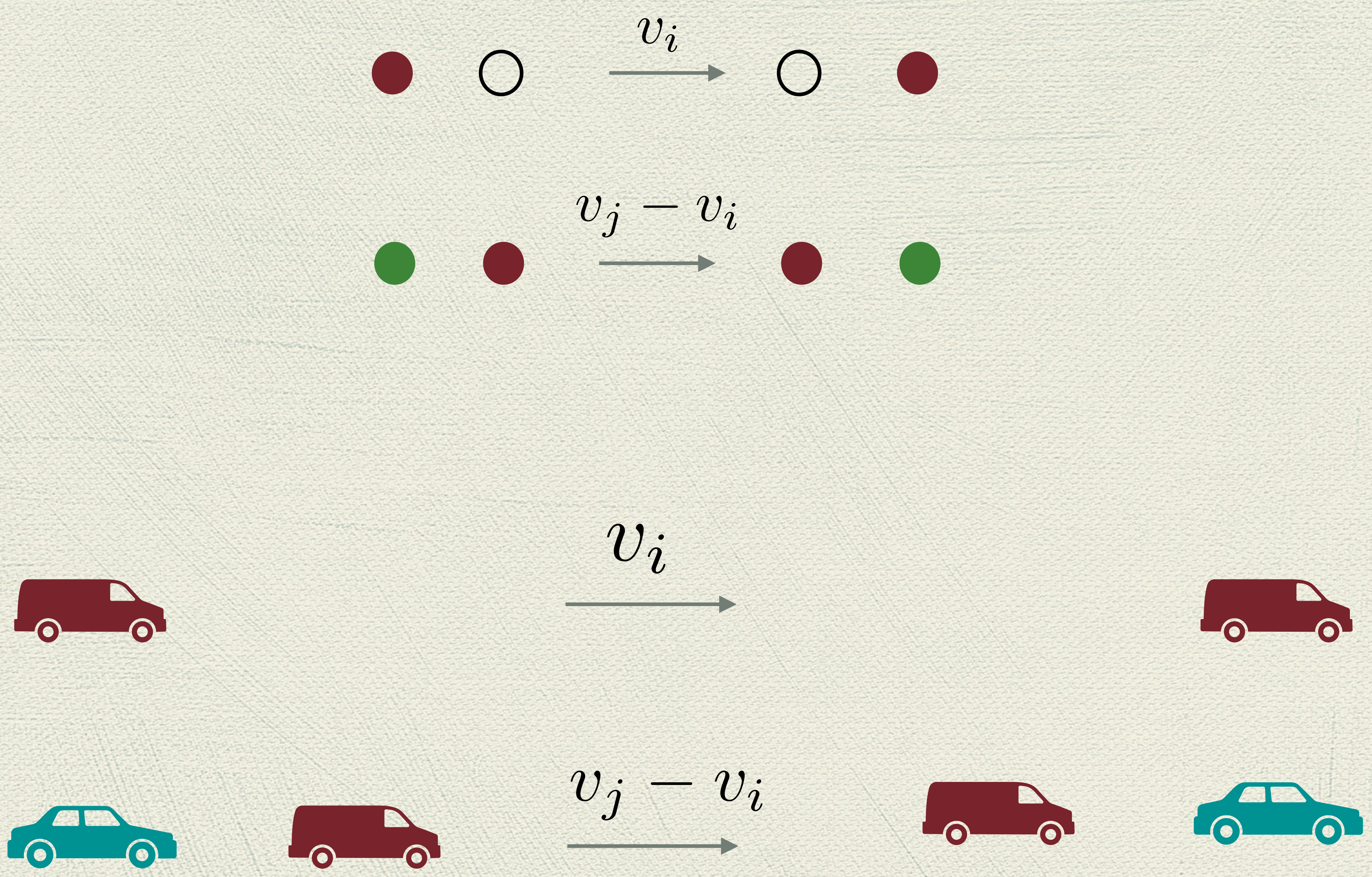
$$v_p \geq v_{p-1} \geq \cdots v_1 \geq 0$$

Multi-species asymmetric simple exclusion process and its relation to traffic flow,
V. Karimipour, Phys. Rev. E, 1999.

Different types of particles



Relation to Traffic Flow

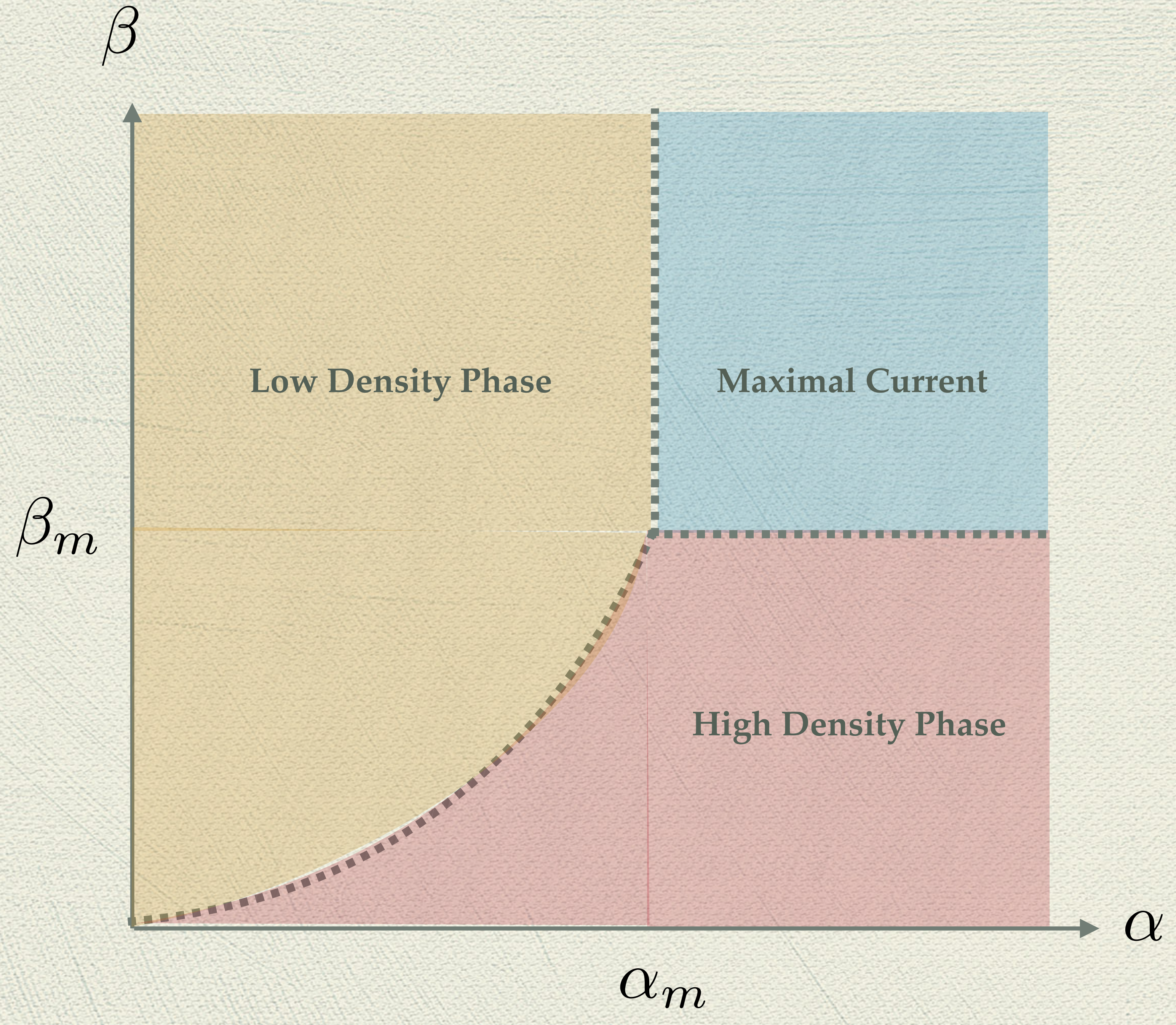


Multi-species asymmetric simple exclusion process and its relation to traffic flow,
V. Karimipour, Phys. Rev. E, 1999.

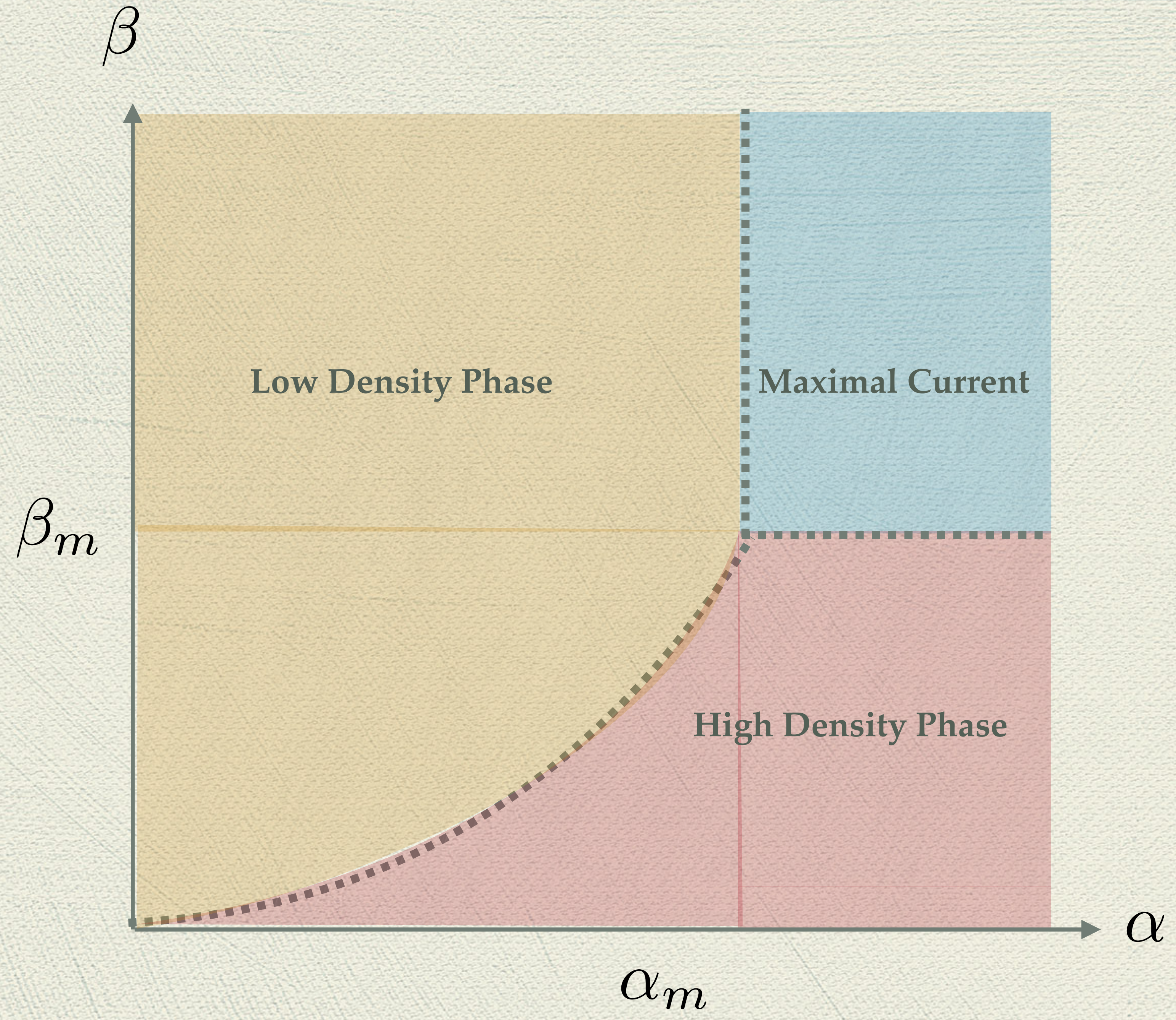
$$\langle v \rangle = \int_0^\infty v P(v) dv$$

$$l[P] := \frac{1}{v_1^2} - \langle \frac{v}{(v - v_1)^2} \rangle$$

$$l[P] < 0$$

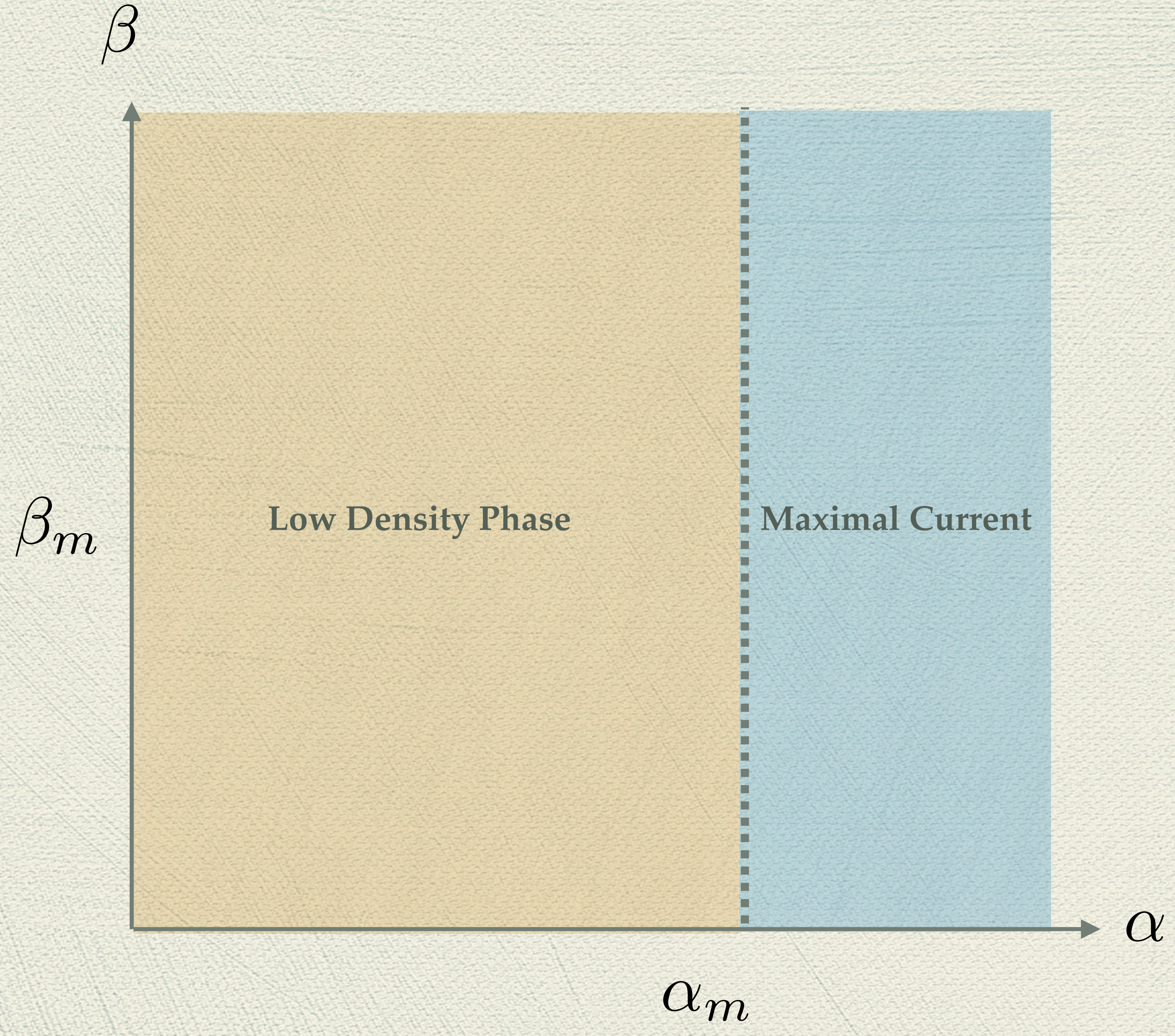


M. Khorrami, V. Karimipour, Jour. Stat. Phys, 2000.



M. Khorrami, V. Karimipour, Jour. Stat. Phys, 2000.

$$l[P] > 0$$

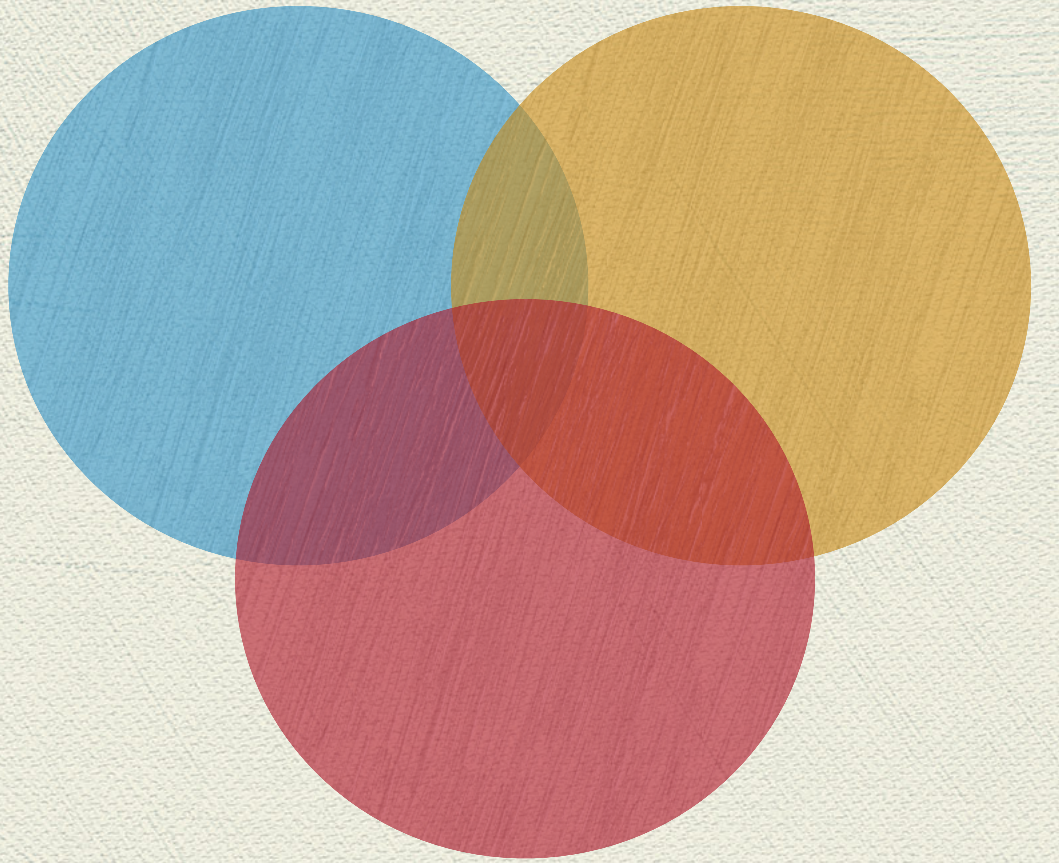


M. Khorrami, V. Karimipour, Jour. Stat. Phys, 2000.

Extension to the quantum domain?

Quantum Information

Condensed Matter Physics



Stochastic Processess

Matrix product representation of all valence bond states
V. Karimipour, L. Memarzadeh, Phys. Rev. B, 2008

Thanks for your attention