

Homoclinic Explosion in a Lorenz Like System

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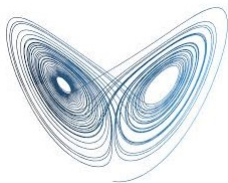
Abstract

The main purpose of this work is to study the transition to Chaos in a Lorenz like system describing the dynamics of cell signaling. By tracing the branch of limit cycles past a limit point with the aid of numerical bifurcation analysis we reveal the existence of a homoclinic bifurcation point. The existence of the symmetric pair of the homoclinic trajectories is proved analytically with the use of undetermined coefficient method. We show the existence of a homoclinic explosion that gives rise first to a preturbulence regime and then to a transition to a stable chaotic regime.

Introduction

Lorenz Like Systems

- Historically important of the Lorenz System.
- Marks the onset of many Lorenz Like Systems (Rossler, Chen, Lu, Multi-Equilibria).
- The analytical and numerical computations in these systems reveal a wide variety of chaotic dynamical behaviours.



$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

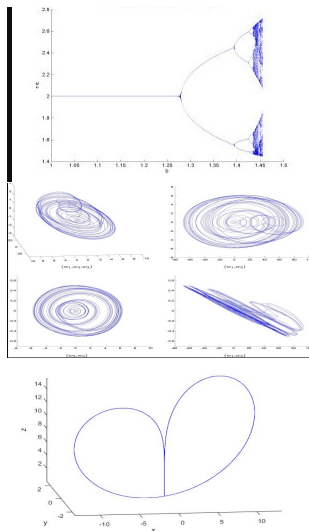
$$\frac{dz}{dt} = xy - \beta z$$

Introduction

Chaotic Dynamics

In general, Chaotic regimes, widely exist in 3d ODEs. Chaos emerges through various mechanisms such as,

- Period Doubling Bifurcation
- Shilnikov Chaos
- Homoclinic Explosion



Introduction

Homoclinic Explosion

- We refer to Homoclinic Explosion as the point that is marked by the Homoclinic Bifurcation in a system that leads to a rearrangement of the systems' manifolds. This suggests the transition to chaos.
- Homoclinic Bifurcation point marks a preturbulent regimes where chaotic dynamics that emerge still remain unstable (Saddle type of Chaos).
- Preturbulence and Saddle type of Chaos are introduced by Kaplan and York.
- This transition leads to a Stable Chaotic Dynamics.

The Lorenz Like System

Tlr4 Protein

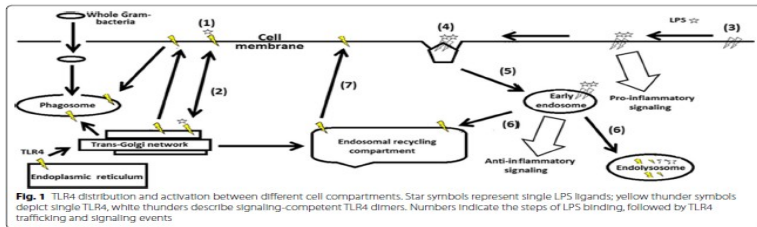
Chaotic Dynamics analysis in this study aims to contribute to the mathematical study of the relation between the intracellular TLR4 oscillations during sepsis progression. The importance of Tlr4 protein lies to,

- The crucial role that it plays in the innate immune system.
- The fact that it is a basic mechanism for Gram-Negative Bacteria recognition.
- The fact that tlr4 oscillations are related to the sepsis progression.

The Lorenz Like System

Tlr4 Protein

- *R., Stan, F., Soriano, M., Camargo, "A Mathematical model relates intracellular Tlr4 oscillations to sepsis progression", <https://doi.org/10.1186/s13104-018-3561-9>*
- Describes the Tlr4 signaling between the various cell compartments.



The Lorenz Like System

Mathematical Model

The 3D System of Ordinary Differential Equations similar to Lorenz System that describes the dynamics of cell signaling.

$$\dot{x}(t) = fx(t) - y(t)z(t)$$

$$\dot{y}(t) = x(t) - cy(t)$$

$$\dot{z}(t) = x(t)y(t) - dz(t)$$

- x, y, z describe the (dimensionless) variations of concentration of TLR4 of a nominal value in various areas of the system.
- f, c, d are the various rates of TLR4 protein between cell compartments, with both negative and positive values.

The Lorenz Like System

Mathematical Model

- Symmetry Equations are invariant under $(x, y, z) \rightarrow (-x, -y, z)$ transformation. The system has a x, y axis symmetry.
- Fixed Points

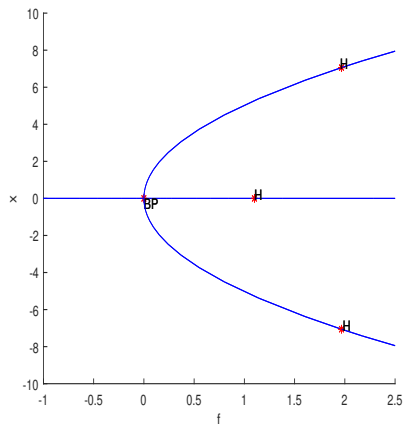
$$(x^*, y^*, z^*) = \begin{cases} E_0 = (0, 0, 0) \\ E_1 = (c\sqrt{fd}, \sqrt{fd}, fc) \\ E_2 = (-c\sqrt{fd}, -\sqrt{fd}, fc) \end{cases}$$

The Lorenz Like System

Mathematical Model

A Pitchfork Bifurcation appears at the origin.

For $d \geq 0$ the Branching Point at the origin gives rise to a symmetric pair of fixed points, while the origin stays as a fixed point.



The Lorenz Like System

Mathematical Model

- From the Jacobian Matrix we take the characteristic polynomial
$$\lambda^3 - p\lambda^2 - q\lambda - m = 0$$
- We use the Cardano method to compute the roots of the cubic equation and the discriminant Δ determines its sign.
- Analytical and Numerical computations converge to each other.
- - ▶ E_0 : Stable Node turns to Saddle Node (for $f > 0$)
 - ▶ E_1 : Stable Focus turns to Saddle Focus (for $f > f_{hopf}$)
 - ▶ E_2 : Stable Focus turns to Saddle Focus (for $f > f_{hopf}$)
- We prove analytically the existence of a Hopf Bifurcation point at
$$f = \frac{c+d}{3}$$

One Parameter Numerical Bifurcation Analysis

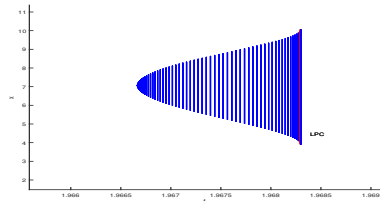
Methodology

- We performed one dimensional bifurcation analysis taking f as our bifurcation parameter.
- The Bifurcation Analysis has been done using MatCont package of Matlab and Newton Raphson algorithm bifurcation analysis was made.
- We focus on the Continuation of Limit Cycles.

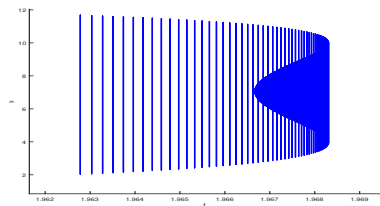
One Parameter Numerical Bifurcation Analysis

Bifurcations Diagrams

One parameter numerical bifurcation of parameter f from f_{hopf} reveals a stable branch of limit cycles.



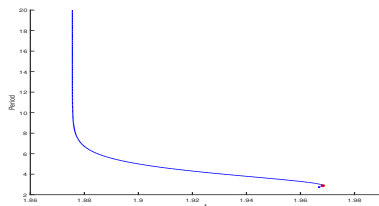
The stable branch of limit cycles turns to unstable at a Limit Point Cycle



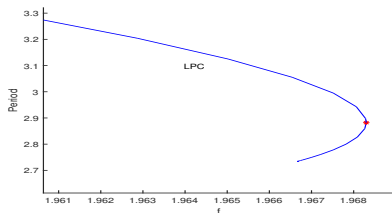
One Parameter Numerical Bifurcation Analysis

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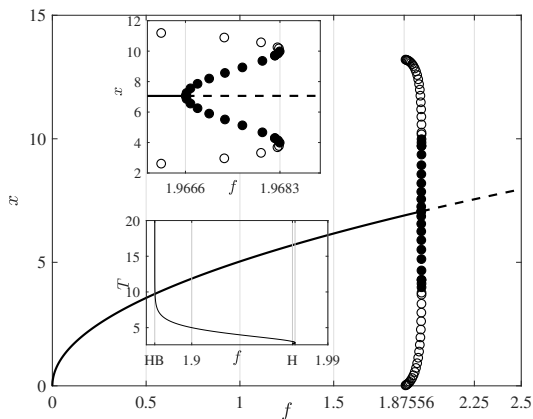


The existence of a Homoclinic Bifurcation at $f_{hom} = 1.87556$



One Parameter Numerical Bifurcation Analysis

Overall Bifurcation Diagram

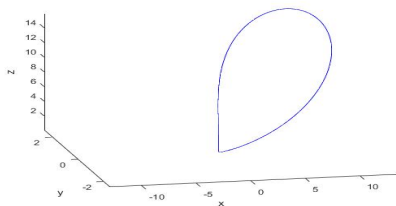


Overall Bifurcation Diagram of one parameter

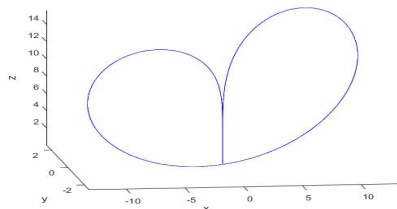
Homoclinic Explosion and Transition to Chaos

Preturbulence Regimes-Saddle Chaos

- Homoclinic Bifurcation point at f_{hom} marks the Homoclinic Explosion to our system.
- We focus on the f values between f_{hom} and $f_{chaotic}$.



The Homoclinic Trajectory at the origin

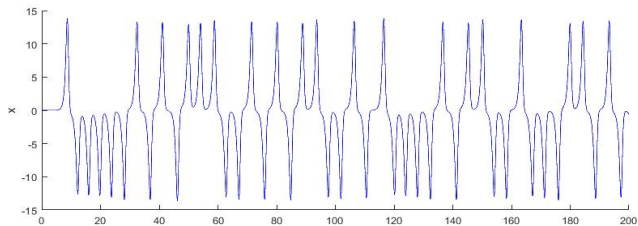
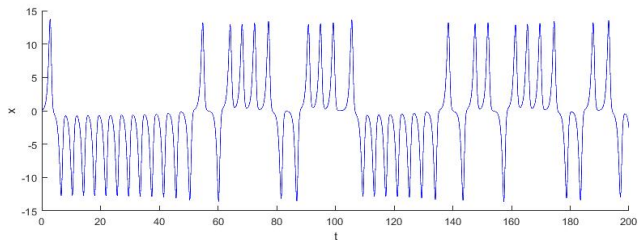


The symmetric pair of homoclinic orbits

Homoclinic Explosion and Transition to Chaos

Preturbulence Regimes-Saddle Chaos

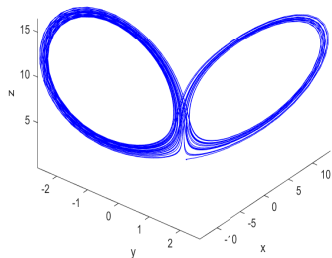
- Homoclinic Explosion signals the transition to chaos via this preturbulence regimes that our system undergoes.



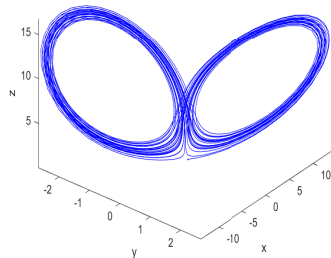
Homoclinic Explosion and Transition to Chaos

Stable Chaos of the System

- The following figures present some characteristic phase portraits around the Hopf bifurcation point.
- The system exhibits a chaotic behavior with two strong chaotic attractors.



$f = 1.945$

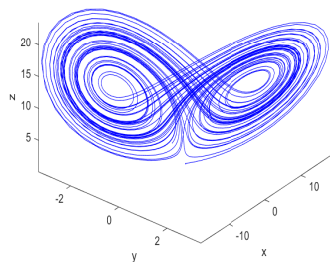


$f = 1.967$

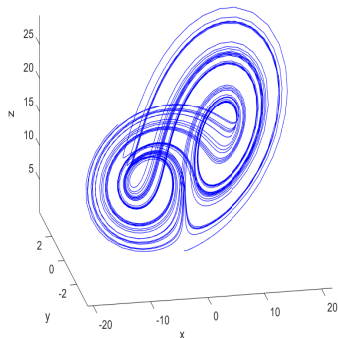
Homoclinic Explosion and Transition to Chaos

Stable Chaos of the System

- This dynamical behaviour is localized for an interval of parameter f approximately $(1.945, 3.1)$.



$f = 2.5$



$f = 2.9$

Homoclinic Explosion and Transition to Chaos

- For $f^\#$ the saddle fixed point of the origin is conditioned a Homoclinic Bifurcation. A Homoclinic Orbit exist at E_0 .
- The initial system transformes into,

$$y[\ddot{y} + \alpha_1\ddot{y} - \alpha_2\dot{y} - \alpha_3y] + \dot{y}[-\ddot{y} + \alpha_4\dot{y} + y^3] + cy^4 = 0$$

- We substitute $y(t) = \sum_{k=1}^{\infty} d_k e^{\mu kt}$, $t > 0$, to determine the semiorbit of homoclinic trajectory
- The method gives us the above coefficients
$$d_{2k} = 0, \quad d_{2k+1} = -\frac{H_k^1 + H_k^2}{H_k^3}$$
- The series converge for $|d_k| < 10^{-k+1} |d_1^k|$.

Conclusion

- We are only at the beginning of understanding and modelling with mathematical tools the complexity of cell biology dynamics that underlies the function of this important family of receptors.
- Our analysis goes further than temporal simulations. It contributes with the bifurcation analysis and the corresponding simulations.
- This is the first study that reveals the mechanisms drive chaotic dynamics that have been observed also in experiments.
- In particular, we prove that the onset of the chaotic regime is due to a homoclinic explosion.
- Two parameter numerical bifurcation analysis is in progress.