Many-body semiclassics for Bose-Hubbard: spectral statistics and random wave approach

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in collaboration with S. Müller, J. D. Urbina, K. Richter

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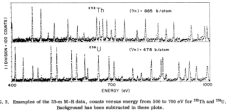
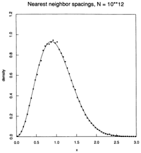
5th of March 2020

Outline

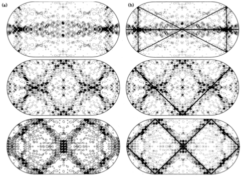
From one-body quantum chaos...



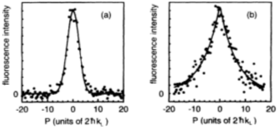
Neutron scattering



Number theory



Quantum dynamics



Cold atoms

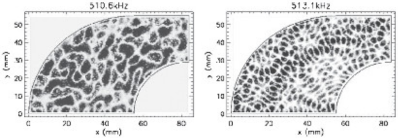
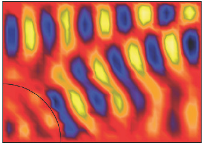


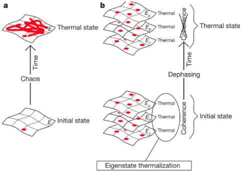
Plate vibration mode



Microwave cavities

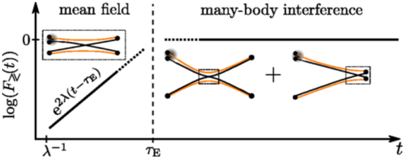
and He atom, quantum dots,. . .

... to very topical (many-body) issues!!



Eigenstate Thermalisation Hypothesis (ETH)

M. Rigol, V. Dunjko, M. Olshanii, Nature **452**, 854 (2008)



Out of time ordered correlators

J. Rammensee, J. D. Urbina, K. Richter, Phys. Rev. Lett. **121**, 124101 (2018)

Signature of chaos beyond spectral statistics, comparison with Random Matrix Theory (RMT)

F. Borgonovi, F. M. Izrailev, L. F. Santos, V. G. Zelevinksy, Phys. Rep. **626**, 1 (2016)
Y. Y. Atas and E. Bogomolny, J. Phys. A **50**, 385102 (2017)

W. Beugeling, A. Bäcker, R. Moessner, M. Haque, Phys. Rev. E **98**, 022204 (2018)

Goal of the talk

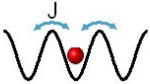
Study of generic quantum many-body systems.

- Spectral statistics
- Eigenvector statistics
- Dynamics

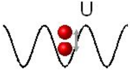
Our model: Bose Hubbard chain

N bosons on a 1D lattice of L sites with onsite disorder.

tunneling J



interaction U



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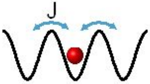
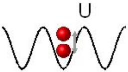


interaction U



$$\hat{H} = \underbrace{-J \sum_{l=0}^{L-1} (\hat{a}_{l+1}^\dagger \hat{a}_l + \hat{a}_l^\dagger \hat{a}_{l+1})}_{\text{hopping}} + \underbrace{U \sum_{l=0}^{L-1} (\hat{a}_l^\dagger)^2 \hat{a}_l^2}_{\text{interaction}} + \underbrace{\sum_{l=0}^{L-1} \epsilon_l \hat{a}_l^\dagger \hat{a}_l}_{\text{onsite potential}}$$

$\hat{a}_l, \hat{a}_l^\dagger$: bosonic operators

ϵ_l: i.i.d. random variable (zero mean, unit variance)

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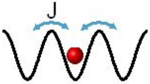

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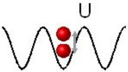

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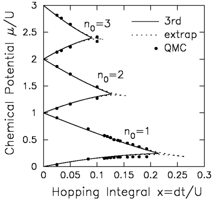
Ground state (i.e. $T = 0$):



Quantum phase transition
(superfluid/Mott insulator)

M. P. A. Fischer, P. B. Weichman, G. Grinstein, and D. S. Fisher, PRB (1989)

J. K. Freericks and H. Monien, EPL (1994)

Ground state statistics: multifractality

J. Lindinger, A. Buchleitner, A. Rodríguez. PRL **122**, 106603 (2019)

Outline

Method

Goal: compute the quantum level density $\rho(E)$

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1. Use the coherent state representation:

$$\hat{a}_l|\psi\rangle = \psi_l|\psi\rangle, \qquad \langle\psi|\,\hat{a}_l^\dagger = \psi_l^*\,\langle\psi| \ .$$

   to write the propagator

$$K(\psi^{(f)\,*}, t|\psi^{(i)}, 0) \equiv \left\langle \psi^{(f)} \middle| e^{-\frac{i}{\hbar}\hat{H}t} \middle| \psi^{(i)} \right\rangle \ .$$

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Insert the closure relation: $1 = \int \frac{\mathrm{d}\overline{\psi}\mathrm{d}\psi}{(2\pi \mathrm{i})^L}\,|\psi\rangle\,\langle\psi|$

$$K(\psi^{(f)\,*}, t|\psi^{(i)}, 0) = \int D[\overline{\psi}, \psi] e^{iR[\overline{\psi},\psi]/\hbar}$$

$\overline{\psi}, \psi$: 2 complex independent coordinates related to Bargmann (or holomorphic) representation of $\hat{H}$.

Semiclassical approximation for Bose Hubbard model

$$K(\psi^{(f)\,*}, t \big| \psi^{(i)}, 0) = \int D[\overline{\psi}, \psi] e^{iR[\overline{\psi},\psi]/\hbar}$$

The very important object is the classical action:

$$R[\overline{\psi}, \psi] \quad = \int_0^t \left[-\frac{\hbar}{\mathrm{i}} \overline{\psi}(\tau)\dot{\psi}(\tau) - \mathcal{H}(\overline{\psi}(\tau), \psi(\tau)) \right] \mathrm{d}\tau$$

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It contains the classical Hamiltonian (which depends of the choice of ordering):

$$\mathcal{H}(\overline{\psi}, \psi) = \langle \overline{\psi} | \hat{H} | \psi \rangle$$

Here (no disorder: $\epsilon_l = 0$):

$$\mathcal{H}(\overline{\psi}, \psi) = -J \sum_{l=0}^{L-1} (\overline{\psi}_{l+1} \psi_l + \overline{\psi}_l \psi_{l+1}) + U \sum_{l=0}^{L-1} \overline{\psi}_l^2 \psi_l^2$$

Semiclassical approximation for Bose Hubbard model

2. Saddle point approximation of the path integral
($\hbar \to 0 \;\Leftrightarrow\; N \to \infty$).

M. Baranger, M. A. M. de Aguiar, F. Keck, H. J. Korsch, B. Schellaaß, J. Phys. A **34**, 7227 (2001)

T. F. Viscondi, M. A. M. de Aguiar, J. Math Phys. **52**, 052104 (2011)

Saddle equations:

$$\mathrm{i}\hbar\dot{\psi} = \frac{\partial \mathcal{H}}{\partial \overline{\psi}}, \quad \mathrm{i}\hbar\dot{\overline{\psi}} = -\frac{\partial \mathcal{H}}{\partial \psi}.$$

$$\mathrm{i}\hbar\dot{\psi}_l = -J\left(\psi_{l-1} + \psi_{l+1}\right) + 2U\overline{\psi}_l\psi_l^2$$

Discrete nonlinear Schrödinger equation: Mean field!

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Discrete nonlinear Schrödinger equation: Mean field!

Suming over all the solutions of the saddle/mean field equations:

$$K(\psi^{(f)\,*}, t|\psi^{(i)}, 0) \approx \sum_p \left| \frac{\mathrm{i}}{\hbar} \frac{\partial^2 R_p}{\partial \psi^{(f)*} \partial \psi^{(i)}} \right|^{1/2} e^{\frac{\mathrm{i}}{\hbar}(R_p + R_p^S) - \mathrm{i}\nu_p \frac{\pi}{2}}$$

R_p: classical action. R_p^S: Solari phase due to the ordering.
ν_p: Maslov index

Semiclassical approximation for Bose Hubbard model

3. Level density is given by:

$$\rho_N(E) = -\frac{1}{\pi} \lim_{\eta \to 0} \mathrm{Im} \int_0^\infty e^{(E+\mathrm{i}\eta)t/\hbar} \mathrm{tr} \left(\delta_{\hat{N},N} K \right) \mathrm{d}t$$

Approximation within the saddle point approximation.

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Integration over zero modes due to conservation of $\hat{N}$.

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Approximation within the saddle point approximation.

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$$\rho_N(E) \approx \overline{\rho}_N(E) + \frac{1}{\pi\hbar}\mathrm{Re}\sum_p \frac{T_p^{\mathrm{prim}} e^{-i\mu_p \frac{\pi}{2}}}{\sqrt{|\det(M_p - 1)|}} e^{iS_p/\hbar}$$

$T_p^{\mathrm{prim}}, \mu_p, M_p, S_p$: quantities obtained from mean field theory

Gutzwiller-like trace formula!

Generalise quantum/classical correspondence for second quantised models

R. Dubertrand and S. Müller, New J. Phys. **18**, 033009 (2016)

T. Engl, J. D. Urbina, K. Richter, Phys. Rev. E **92**, 062907 (2015)

Semiclassical correspondence for Bose Hubbard model: interpretation

The mean field solutions are analogous to classical solutions.

Gutzwiller trace formula to RMT spectral statistics: √

Semiclassical correspondence's perspective:

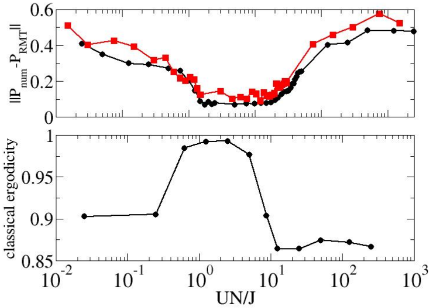
Spectral statistics close to Random Matrix Theory

using

$P(r), \; r = (E_{n+1} - E_n)/(E_n - E_{n-1})$.

⇑

Ergodic mean field solutions



Quantitative criterion to apply RMT.

Outline

Berry's ansatz for 1− body problem

2−dimensional quantum billiard

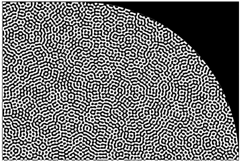
$$\hat{H}\,|\psi_j\rangle = E_j\,|\psi_j\rangle$$

For $j \gg 1$, $\psi_j(\mathbf{r})$ looks like a Gaussian random field,

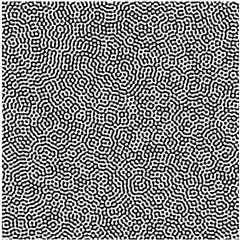
$$\psi_j(\mathbf{r}) \approx \sum_{p=0}^{\infty} a_p J_p(kr)\cos(p\theta), \quad E_j = \hbar^2 k^2/2m\,.$$

where a_p are i.i.d. Gaussian random variables.
Induces amplitude correlations:
$\langle \psi(\mathbf{r})\psi(\mathbf{r}')\rangle = J_0(k|\mathbf{r}-\mathbf{r}'|)$

M. V. Berry, J. Phys. A, **10** 2083 (1977)



$\approx 10,000^{\text{th}}$ Eigenstate of a stadium billiard



Sample of a random wave

How to lift Berry's ansatz for many-body problems?

$$\hat{H}\,|\Psi_j\rangle = E_j\,|\Psi_j\rangle$$

Eigenstate expansion in the Fock state basis: $|\mathbf{n}\rangle = |n_1, \ldots, n_L\rangle$:

$$|\Psi_j\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}}^{(j)} |\mathbf{n}\rangle$$

$$c_{\mathbf{n}} \leftrightarrow a_p$$

Statistical description of the $c_{\mathbf{n}}^{(j)}$'s in the bulk???

Berry's ansatz at the basis of ETH...

Berry's ansatz for many-body

Criteria for applicability:

- semiclassical regime: large *N* ($\hbar_{\mathrm{eff}} = 1/N$)$\surd$
- chaotic classical limit:
  RMT spectral statistics and ergodic mean field solutions $\surd$

Berry's ansatz for many-body

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- semiclassical regime: large N ($\hbar_{\rm eff} = 1/N$)$\surd$
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Berry's ansatz predictions in Fock space:

1. $$|\Psi_j\rangle \approx \sum_{\mathbf{n}} c_{\mathbf{n}} |\mathbf{n}\rangle$$

   $c_{\mathbf{n}}$ random variables with Gaussian joint probability distribution

   $$P(\mathbf{c} = \{c_{\mathbf{n}}\}) \propto \exp\left(-\mathbf{c}^T R^{-1} \mathbf{c}\right)$$

2. Quantum/classical correspondence for many-body:
   Analytical prediction for $R(\mathbf{m}, \mathbf{n})$

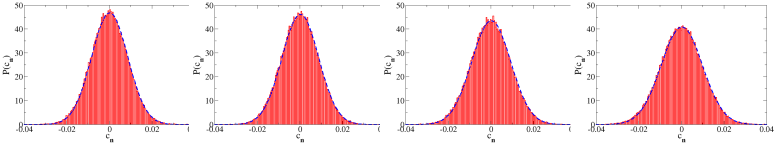
RMT prediction: $P(\mathbf{c} = \{c_{\mathbf{n}}\}) \propto \delta\left(1 - \mathbf{c}^T \mathbf{c}\right)$

Numerical check of Berry's ansatz: diagonal part

Method:

- Numerical Diagonalisation to get E_j, $|\Psi_j\rangle$
- Average over disorder (random variables ϵ_l)

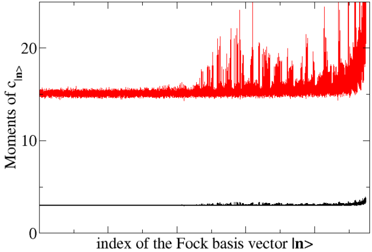
1. Gaussian distribution for a fixed $|\mathbf{n}\rangle$
   $N = 20$, $L = 5$, $UN/L = 2$.



$|4\ 4\ 4\ 4\ 4\rangle$ $|4\ 5\ 4\ 3\ 4\rangle$ $|7\ 3\ 0\ 3\ 7\rangle$ $|10\ 2\ 3\ 3\ 2\rangle$

Numerical check of Berry's ansatz: diagonal part

2. Compute the higher moments of the distribution $P(\{c_{\mathbf{n}}\})$
   $N = 25$, $L = 5$, $\dim \mathcal{H} = 23751$



$2k^{\text{th}}$ moment:

$$m_{2k} = \frac{\langle c_{\mathbf{n}}^{2k} \rangle}{\left(\langle c_{\mathbf{n}}^{2} \rangle\right)^{k}}$$

Gaussian

$$m_{2k} = 2^k \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$m_4 = 3$$

$$m_6 = 15$$

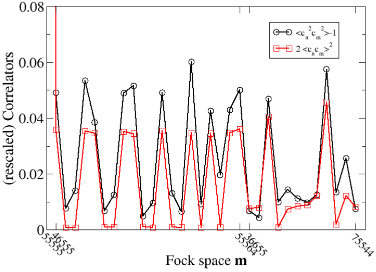
Similar results in spin chains

Y. Y. Atas and E. Bogomolny, J. Phys. A **50**, 385102 (2017);

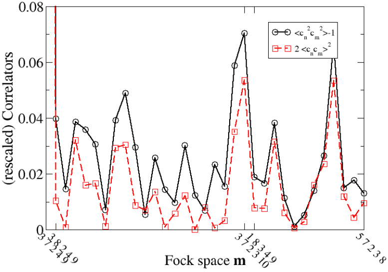
W. Beugeling, A. Bäcker, R. Moessner, M. Haque, Phys. Rev. E **98**, 022204 (2018)

3. The Gaussian character of the joint distribution leads to:

$$\left\langle c_m^2 c_n^2 \right\rangle - \left\langle c_m^2 \right\rangle \left\langle c_n^2 \right\rangle = 2 \left(\left\langle c_m c_n \right\rangle \right)^2$$



n = 5 5 5 5 5



n = 3 7 2 4 9

Correlations in Fock space: theory

4. Analytical prediction for the 2−point correlator

$$\langle c_{\mathbf{m}} c_{\mathbf{n}} \rangle \propto R(\mathbf{m}, \mathbf{n})$$

with

$$R(\mathbf{m}, \mathbf{n}) \equiv \int \mathrm{d}E \, f_\Delta(E - E_0) \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^L} \delta \left[E - H \left(\frac{\mathbf{m} + \mathbf{n}}{2}, \boldsymbol{\theta} \right) \right] e^{\mathrm{i} \sum_{l=0}^{L-1} (m_l - n_l) \theta_l}$$

where $f_\Delta(E - E_0)$ is the spectral density centered at $E = E_0$ and of width Δ.

Correlations in Fock space: theory

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Insert the classical symbol for the Hamiltonian:

$$H(\mathbf{m},\theta) = -J\sum_{l=0}^{L-1} \sqrt{m_l m_{l+1}} \cos(\theta_l - \theta_{l+1}) + U \sum_{l=0}^{L-1} m_l(m_l - 1) + \sum_{l=0}^{L-1} \epsilon_l m_l$$

For periodic BC the angular integral can be computed recursively.

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For periodic BC the angular integral can be computed recursively. Eventually

$$R(\mathbf{m},\mathbf{n}) = \int \mathrm{d}E\, f_\Delta(E-E_0) \sum_{Q\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{2\pi} e^{\mathrm{i}\tau[E - U\sum_l I_l(I_l-1) - \sum_l \epsilon_l I_l]} \times$$

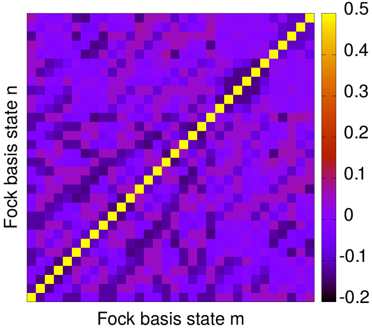
$$\prod_{l=0}^{L-1} \left[e^{\mathrm{i}[\sum_{k=1}^{l}(n_k - m_k) + Q]\pi/2} J_{\sum_{k=1}^{l}(n_k-m_k)+Q}(J\tau\sqrt{I_l I_{l+1}}) \right]$$

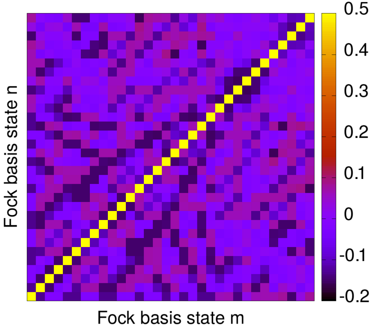
with $I_l = \frac{m_l + n_l}{2}$.

To be averaged over the random variables ϵ_l.

2−point correlation: theory vs numerics

Checking the numerics: plotting $\frac{\langle c_m c_n \rangle}{\sqrt{\langle c_m^2 \rangle \langle c_n^2 \rangle}}$ for varying ***m*** and ***n***



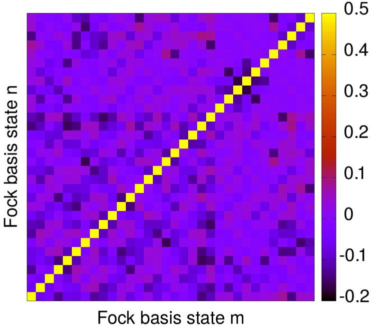


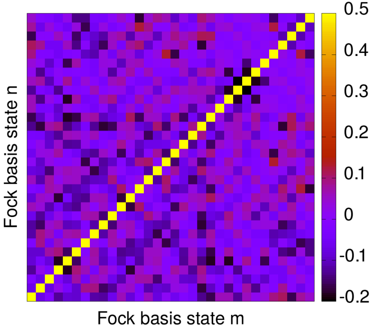
Theory

States 'close to' $\mathbf{n} = 5\ 5\ 5\ 5\ 5$

Numerics

2−point correlation: theory vs numerics





Theory

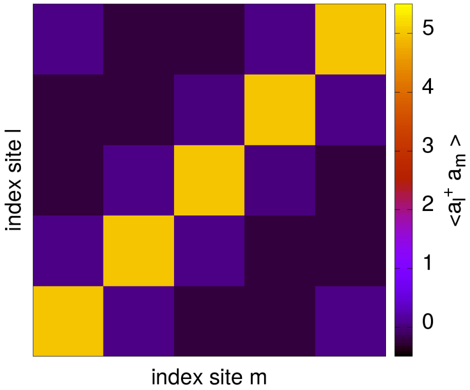
States 'close to' $\mathbf{n} = 3\ 7\ 2\ 4\ 9$

Numerics

R. Dubertrand, J. D. Urbina, K. Richter, in prep.

Macroscopic cross-correlations: 1−body observable

$N = 25, L = 5$. Average over the eigenstates in the bulk of $\left\langle a_l^\dagger a_m \right\rangle$



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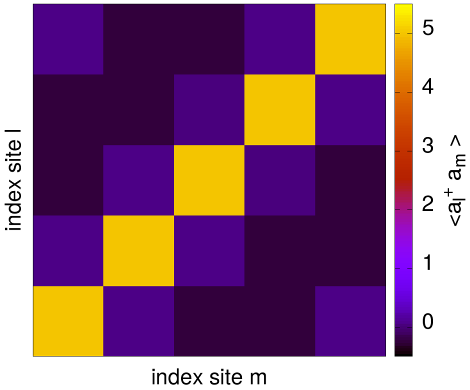

Consistent with ETH.

Outline

Quantum/classical correspondence can be generalised for many-body systems.

Powerful predictions:

- spectral statistics:
  can justify RMT and predict when to apply it. Measure from ergodic to non ergodic models.
- eigenstate statistics in the bulk
  Berry's ansatz for many-body eigenstates, how to go **beyond RMT**

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For the future...

- quantitative understanding of thermalisation
- $d \to \infty$: field theory