STEADY STATES AND TRAVELING WAVES OF HEISENBERG SPIN SYSTEMS

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SUMMARY

In a recent publication [1], we have obtained analytically expressions for the equilibrium states and traveling wave solutions of:

- The Heisenberg and 1+1 system in the form

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx},$$

- and the Myrzakulov - I 2+1 spin system

$$\vec{S}_t = (\vec{S} \times \vec{S}_y + u\vec{S})_x, \quad u_x = -(\vec{S}, \vec{S}_x \times \vec{S}_y), \quad \vec{S} = (S_1, S_2, S_3),$$
$$S_1^2 + S_2^2 + S_3^2 = 1.$$

Here, we summarize these findings and also exhibit analogous results for:

- The Isotropic LLG Heisenberg system with Gilbert damping

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \lambda(\vec{S}_{xx} - \vec{S} \cdot \vec{S}_{xx})\vec{S})$$

METHODS AND RESULTS

- Regarding the steady states, which are time independent, we set the LHS of these equations equal to zero, write our spin vector in the form $\vec{S} = (u(x), v(x), w(x))$, where $u^2 + v^2 + w^2 = 1$, and obtain systems of second order ODES in these variables.
- In the cases of the Heisenberg and M - I spin systems, we show that these equations can be directly solved by trigonometric functions, while in others, we use the fact that they possess in Painlevé property to also arrive at similarly simple solutions.
- Seeking traveling wave solutions, we replace $\vec{S}(x, t)$ by $\vec{S}(x + \mu t)$ and solve similar ODEs where the independent variable now is $\xi = x + \mu t$, μ being the velocity of the wave.
- To solve the isotropic LLG equation, we adopt a similar approach and find simple curves on the unit sphere which represent stable attractors since this equation involves Gilbert damping and is in fact be equivalent to a damped NLS equation [4-5].

Introduction

More than 50 years ago, it was discovered that there exist nonlinear PDEs in 1+1 dimensions that are completely integrable, in the sense that they are linearizable by the Inverse Scattering Transform and Lax pair formalisms and possess infinitely many conservation laws [?]. Some of the most famous such equations were the Korteweg de Vries (KdV) equation, the Modified KdV equation, the sine–Gordon and the Nonlinear Schrödinger (NLS) equation.
In the late 1990's, a new class of completely integrable spin models emerged, mostly in 2+1 dimensions, to which the Inverse Scattering, Lax pairs, Bäcklund transformations,N-soliton solutions, bilinear forms, etc. could be applied [10-11]. Remarkably enough, all these spin models, from Heisenberg ferromagnets to a new class of so–called Myrzakulov equations were shown to be equivalent to PDE systems of the NLS type [11-12].

STEADY STATES OF THE HEISENBERG SPIN SYSTEM

Setting $\vec{S}_t = 0$ and using $\vec{S} = (u, v, w)$ we obtain from (1) after one integration the system of ODEs:

$$uv' - u'v = c_1, \quad uw' - u'w = c_2, \quad wv' - w'v = c_3, \tag{1}$$

where the $c_i, i = 1, 2, 3$ are arbitrary constants. Since these equations are not independent, we combine them and obtain the condition

$$-c_1 w + c_2 v + c_3 u = 0. \tag{2}$$

the above imply that the fixed points we are seeking lie on maximal circles formed by the intersection of the sphere $u^2 + v^2 + w^2 = 1$ and the plane (2).

In fact, it is possible to obtain a parametrization of these circles by a single angular parameter. For example, let $c_2 = 0$. Then, from (2) we have that $c_1 w = c_3 u$, whence the sphere condition gives

$$u^2 + v^2 + (\frac{c_3}{c_1})^2 = 1, \quad \Rightarrow v = \pm\sqrt{1 - C^2 u^2}, \quad C^2 = 1 + (\frac{c_3}{c_1})^2. \qquad (3)$$

If we now set $Cu = sin\theta$, we finally have:

$$u = \frac{c_1}{\sqrt{c_1^2 + c_3^2}} sin\theta, \quad \Rightarrow v = cos\theta, \quad w = \frac{c_3}{\sqrt{c_1^2 + c_3^2}} sin\theta \qquad (4)$$

Thus, different curves of fixed points can been determined on the unit sphere. These fixed points, of course, are not isolated and are linearly stable upon small changes of their coordinates.

TRAVELING WAVES OF THE HEISENBERG SYSTEM

Assuming that our spin variables depend on the single wave variable $\xi = x + \mu t$, $\vec{S} = (u, v, w)$ now satisfies $\mu \vec{S}' = (\vec{S} \times \vec{S}')'$, where primes denote differentiation with respect to ξ and consequently after one integration we get

$$vw' - v'w = \mu u + c_1, \quad -uw' + u'w = \mu v + c_2, \quad uv' - vu' = \mu w + c_3 \tag{5}$$

Using spherical coordinates on the unit sphere:

$$u = cos\theta sin\varphi, \quad v = sin\theta sin\varphi, \quad w = cos\varphi. \tag{6}$$

we substitute in the above equations and after some manipulations we arrive at the equations

$$\varphi'' = (-c_1 cos\theta - c_2 sin\theta), \quad \theta' = (\frac{\mu + c_3 cos\varphi}{sin\varphi})(\frac{\mu cos\varphi + c_3}{sin^2\varphi}) = -\frac{dV}{d\varphi} \tag{7}$$

These equations can be integrated explicitly to find the function $V(\varphi)$.

Hence the second integral that solves the system (7) is:

$$\frac{1}{2}\varphi'^2 + V(\varphi) = E = const.$$

Letting $w = cos\varphi$ allows us to finally show that

$$w = cos\varphi = -c_3\mu/2E \pm Asin(\sqrt{2E}(x + \mu t) + C) \quad (8)$$

Solving the corresponding equation for $sin\theta(t)$ as follows:

$$sin\theta = \frac{c_1 A\sqrt{2E} cos(\eta) \pm c_2\{(c_1^2 + c_2^2)sin^2\varphi - 2EA^2 cos^2(\eta)\}^{1/2}}{(c_1^2 + c_2^2)sin\varphi} \quad (9)$$

we obtain it again in the form of a simple trigonometric function in $\eta = \sqrt{2E}(x + \mu t) + C$.

We write our unknown spin vector as $\vec{S}=(X,Y,Z), X^2+Y^2+Z^2=1$, where all variables are functions of x,y,t. To find fixed points we must solve the equation $(\vec{S}\times\vec{S}_y+u\vec{S})_x=0 \Rightarrow \vec{S}\times\vec{S}_y+u\vec{S}=\vec{F}(y)$ with $F(Y)$ arbitrary. For simplicity we assume $\vec{S}=\vec{S}(y), u=u(y)$ which implies that the second $M-I$ equation is automatically satisfied and set $F=(1,1,1)$. Defining $u=\frac{Z}{Y}, v=\frac{X}{Y}$ to obtain the two equations

$$u'+v'+v-u=u^2-v^2, \quad u'-v'+v+u=2+(u-v)^2. \tag{10}$$

while introducing the new variables, $S=u+v, D=u-V$ finally leads to single second order ODE:

$$D''=3DD'-3D-D^3 \tag{11}$$

which can be shown to be completely integrable using the Painlev$\acute{e}$ property [13] and is solved by simple trigonometric functions in y, $w=Acos\sqrt{3}y+Bsin\sqrt{3}y$.

We may now solve for Y from the expression:

$$\frac{1}{2Y^2}\left(\frac{E - Asin\psi + Bcos\psi}{E}\right)^2 + \frac{1}{2}\frac{A^2+B^2}{E^2} = \tag{12}$$

$$= \frac{6C^2+A^2+B^2}{2E^2} \Rightarrow Y = \left(\frac{2}{3}\right)^{1/2}\frac{\pm E}{\sqrt{6C^2+A^2+B^2}}$$

and finally obtain explicit expressions for X, Z as well

$$Z = K(-(A + B\sqrt{3})sin\psi) + (B - A\sqrt{3})cos\psi + 2C\sqrt{3}$$

and

$$X = K(-(A - B\sqrt{3})sin\psi) + (B + A\sqrt{3})cos\psi + 2C\sqrt{3}$$

in terms of simple trigonometric functions, where K is the constant $K = \frac{\pm 1/\sqrt{6}}{\sqrt{6C^2+A^2+B^2}}$.

STEADY STATES FOR THE ISOTROPIC LLG EQUATION

The Isotropic LLG equation for Heisenberg spins in the presence of Gilbert damping [?] is

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \lambda(\vec{S}_{xx} - \vec{S} \cdot \vec{S}_{xx})\vec{S}) \tag{13}$$

where λ may be thought as a small parameter. We substitute again here $\vec{S} = (u, v, w)$ and with $vecS_t = 0$ obtain the system

$$0 = vw'' - wv'' + \lambda u' - \lambda u(uu'' + vv'' + ww'')$$

$$0 = -uw'' + wu'' + \lambda v' - \lambda v(uu'' + vv'' + ww'')$$

$$0 = uv'' - vu'' + \lambda w' - \lambda w(uu'' + vv'' + ww'')$$

Note that this is a homogeneous system of linear equations for u'', v'', w'', which, for nontrivial solutions, requires that the associated determinant vanish:

$$D = (\lambda - \lambda^3)(u^2v^2 + v^2w^2 + u^2w^2 - u^2v^2w^2) = 0 \tag{14}$$

This provides us with a necessary condition for nontrivial equilibrium states of eq. (13) to exist:

$$u^2v^2 + v^2w^2 + u^2w^2 - u^2v^2w^2 = 0 \tag{15}$$

Now, we proceed to integrate the above system of equations for u'', v'', w'' and find, after some calculations that the solutions reduce to the remarkably simple result $v = Aw$, A being an arbitrary constant! Combining this (14) above leads to a family of curves of fixed points, described in the u, v plane by the formula:

$$u^2v^2 = A^2u^2 + v^2 \tag{16}$$

These represent a family of hyperbolas in 3-dimensional space and the steady states lie on their intersection with $u^2 + v^2 + w^2 = 1$.

TRAVELING WAVES FOR THE ISOTROPIC LLG EQUATION

Finally, let us return to the original system of ODEs, for u, v, w, which now become

$$\mu u' = vw'' - wv'' + \lambda u' - \lambda u(uu'' + vv'' + ww'')$$
$$\mu v' = -uw'' + wu'' + \lambda v' - \lambda v(uu'' + vv'' + ww'')$$
$$\mu w' = uv'' - vu'' + \lambda w' - \lambda w(uu'' + vv'' + ww'')$$

where the derivatives now are w.r.t. $\xi - x = \mu t$, μ being the velocity of the wave. Assuming that condition (14) does not hold, we solve these equations as a linear system for u'', v'', w'' and obtain

$$u'' = \frac{D_1}{D}, \quad v'' = \frac{D_2}{D}, \quad v'' = \frac{D_3}{D}, \tag{17}$$

D_1, D_2, D_3 being the usual Kramer 3x3 determinants of the system.

The remarkably simple result that emerges after some tedious algebra is:

$$u'' = 0, \quad v'' = 0, \quad v'' = 0, \tag{18}$$

Thus we arrive at the solutions:

$$u(\xi) = a_1\xi + a_2, \quad v(\xi) = b_1\xi + b_2, \quad w(\xi) = c_1\xi + c_2 \tag{19}$$

for arbitrary constants $a_i, b_i, c_i,\ i = 1, 2$. Combining these equations yields a family of 3-dimensional planes

$$u + v + aw = b \tag{20}$$

for arbitrary a, b, whose intersections with the unit sphere $u^2 + v^2 + w^2 = 1$ provide the location where traveling waves of the LLG lie depending on the initial conditions.

Conclusions and Outlook

- For the Heisenberg spin system we found that the steady states form sinusoidal curves on the unit sphere, while the traveling waves are simple combinations of trigonometric functions of the wave variable $\xi = x + \mu t$.
- For the M - I 2+1 equations, we derived a class of steady states that are given in terms of sines and cosines of the y variable.
- The M - I traveling waves in one variable, are the same as those of the Heisenberg case. However, if we seek M - I waves in two variables, with different velocities in the x and y directions, we arrive at linear PDEs, which possess a wide variety of solutions depending on two arbitrary functions.
- A similar analysis of the Heisenberg isotropic LLG spin system, with Gilbert damping, leads to steady states and traveling waves that form simple curves on the unit sphere. These curves represent stable attractors, and would be interesting to investigate further, in terms of their basin of attraction. They also are independent of λ, except for the value $\lambda = 1$, which needs further investigation.

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